

Selected Problem Solutions from Chapter 1

Problem 1.1

(a) First, we find the cumulative distribution function of $\mathbf{y}$

$$\begin{aligned} F_{\mathbf{y}}(y) &= \Pr\{\mathbf{y} \leq y\} \\ &= \Pr\{\max\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\} \leq y\} \\ &= \Pr\{\mathbf{x}_1 \leq y, \mathbf{x}_2 \leq y, \dots, \mathbf{x}_n \leq y\} \\ &= \Pr\{\mathbf{x}_1 \leq y\} \Pr\{\mathbf{x}_2 \leq y\} \cdots \Pr\{\mathbf{x}_n \leq y\} \\ &= F_{\mathbf{x}}^n(y) \end{aligned} \quad (1)$$

The density function is the derivative of this, namely

$$p_{\mathbf{y}}(y) = \frac{\partial}{\partial y} F_{\mathbf{y}}(y) = n F_{\mathbf{x}}^{n-1}(y) p_{\mathbf{x}}(y) \quad (2)$$

(b) For $p(x) = e^{-x}$, $x \geq 0$, it follows that

$$F(x) = \int_0^x e^{-u} du = 1 - e^{-x} \quad (3)$$

and

$$p(y) = n(1 - e^{-y})^{n-1} e^{-y} \quad (4)$$

Problem 1.3

(a) We follow a number of steps in finding $p(x, y)$ in terms of $p(u_1, u_2)$. First, we find the Jacobean, defined as the determinant of the matrix

$$\begin{bmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} \\ \frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial u_2} \end{bmatrix}.$$

Substituting in the appropriate terms and simplifying, we find that the determinant of this matrix is

$$-\frac{2\pi\sigma^2}{u_1}.$$

We next find the solutions of u_1 and u_2 in terms of x and y . It is straightforward to show that these solutions are

$$u_1 = \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) = g(x, y) \quad (5)$$

and

$$u_2 = \frac{1}{2\pi} \tan^{-1}\left(\frac{y}{x}\right) \quad (6)$$

Finally, by definition,

$$p(x, y) = \frac{u_1}{2\pi\sigma^2} p(u_1, u_2) \Big|_{u_1=g(x,y)} \quad (7)$$

which is

$$p(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) \quad (8)$$

or a two-dimensional Gaussian density function.

(b) It is clear from Eq. 8 that $p(x, y)$ can be written as

$$p(x, y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right) = p(x)p(y). \quad (9)$$

That is, this transformation produces two independent Gaussian random variables.

Problem 1.5

In general, of course, the k^{th} moment of $\mathbf{n}$ is defined to be

$$E\{\mathbf{n}^k\} = \int_0^\infty n^k p(n) dn = \int_0^\infty \frac{(\theta - 1)\gamma^{\theta-1} n^{k+1}}{[n^2 + \gamma^2]^{(\theta+1)/2}} dn \quad (10)$$

If we try to solve this equation by substitution¹ of the variable n with $\gamma \tan(\alpha)$, with the corresponding $dn = \gamma \sec^2(\alpha)$, then Eq. 10 becomes

$$E\{\mathbf{n}^k\} = \int_0^{\pi/2} \frac{(\theta - 1)\gamma^{\theta-1} \gamma^{k+1} \tan^{k+1}(\alpha)}{[\gamma^2 \tan^2(\alpha) + \gamma^2]^{(\theta+1)/2}} \gamma \sec^2(\alpha) d\alpha \quad (11)$$

For $\theta = 4$ and $k = 1$, the mean is

$$E\{\mathbf{n}\} = 3\gamma \int_0^{\pi/2} \sin^2(\alpha) \cos(\alpha) d\alpha = \gamma \quad (\text{for } \theta = 4) \quad (12)$$

and the second moment is

$$E\{\mathbf{n}^2\} = 3\gamma^2 \int_0^{\pi/2} \sin^3(\alpha) d\alpha = 2\gamma^2 \quad (\text{for } \theta = 4) \quad (13)$$

It follows that the variance is

$$V\{\mathbf{n}\} = \gamma^2 \quad (\text{for } \theta = 4) \quad (14)$$

The corresponding results for $\theta = 5$ are

$$E\{\mathbf{n}\} = 4\gamma \int_0^{\pi/2} \sin^2(\alpha) \cos^2(\alpha) d\alpha = \frac{\pi\gamma}{4} \quad (\text{for } \theta = 5), \quad (15)$$

$$E\{\mathbf{n}^2\} = 4\gamma^2 \int_0^{\pi/2} \sin^3(\alpha) \cos(\alpha) d\alpha = \gamma^2 \quad (\text{for } \theta = 5), \quad (16)$$

and

$$V\{\mathbf{n}\} = \left(1 - \frac{\pi^2}{16}\right) \gamma^2 \quad (\text{for } \theta = 5). \quad (17)$$

Problem 1.6

Clearly,

$$E\{[\mathbf{y} - g(\mathbf{x})]^2\} = E\{\mathbf{y}^2\} - 2E\{g(\mathbf{x})\mathbf{y}\} + E\{g^2(\mathbf{x})\} \quad (18)$$

We must calculate the second term. Consider the conditional expectation $E\{g(\mathbf{x})\mathbf{y}|x\}$. This can be written as

$$E\{g(x)\mathbf{y}|x\} = g(x)E\{\mathbf{y}|x\}$$

¹This is a change of variable to solve an integral; not a *statistical* change of variable.

because, conditioned on x , $g(x)$ is a constant. We can then write

$$E\{g(\mathbf{x})\mathbf{y}\} = E\{E\{g(\mathbf{x})\mathbf{y}|x\}\} = E\{g(x)E\{\mathbf{y}|x\}\} = E\{g^2(\mathbf{x})\} \quad (19)$$

and the desired result follows from Eq. 18.

Problem 1.7

By definition, the joint pdf of $\mathbf{x}$ and $\mathbf{y}$ is

$$p(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) \quad (20)$$

and we have the change of random variable defined by the two equations

$$\mathbf{u} = \mathbf{x} \cos \theta - \mathbf{y} \sin \theta \quad (21)$$

and

$$\mathbf{w} = \mathbf{x} \sin \theta + \mathbf{y} \cos \theta \quad (22)$$

It follows that the Jacobian is unity for any θ . The inverse functions are

$$\mathbf{x} = \mathbf{u} \cos \theta + \mathbf{w} \sin \theta \quad (23)$$

and

$$\mathbf{y} = -\mathbf{u} \sin \theta + \mathbf{w} \cos \theta \quad (24)$$

Substituting these last two equations into the joint pdf of $\mathbf{x}$ and $\mathbf{y}$ results in the joint pdf of $\mathbf{u}$ and $\mathbf{w}$ as

$$p(u, w) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{u^2 + w^2}{2\sigma^2}\right) \quad (25)$$

Problem 1.8

Since the mean is zero, the joint characteristic function is

$$\Phi(\omega_1, \omega_2, \omega_3, \omega_4) = \exp\left(-\frac{1}{2}\vec{\omega}^T R \vec{\omega}\right), \quad (26)$$

where $\vec{\omega} = [\omega_1, \dots, \omega_4]^T$ is a column vector of variables and R is a 4×4 autocorrelation matrix whose i^{th} row and j^{th} column is μ_{ij} . We know we can find the first moment of the first component by differentiating $\Phi(\vec{\omega})$ by ω_1

$$\frac{\partial}{\partial \omega_1} \Phi(\vec{\omega}) = \Phi(\vec{\omega}) \left[2 \left(-\frac{1}{2} \right) \vec{m}_1^T \vec{\omega} \right] \quad (27)$$

where $\vec{m}_i^T = [\mu_{i1}, \mu_{i2}, \mu_{i3}, \mu_{i4}]$. To find the second moment such as $E\{\mathbf{x}_1 \mathbf{x}_2\}$, we start by taking the second partial derivative

$$\frac{\partial^2}{\partial \omega_1 \partial \omega_2} \Phi(\vec{\omega}) \cdot (-j)^2$$

Note that

$$\frac{\partial}{\partial \omega_j} [\vec{m}_i^T \vec{\omega}] = \mu_{ij} = \mu_{ji} \quad (28)$$

so that

$$\frac{\partial^2}{\partial \omega_1 \partial \omega_2} \Phi(\vec{\omega}) = \frac{\partial}{\partial \omega_2} [-\vec{m}_1^T \vec{\omega} \Phi(\vec{\omega})]$$

$$= \Phi(\vec{\omega}) [-\mu_{12} + \vec{m}_1^T \vec{\omega} \vec{m}_2^T \vec{\omega}] . \quad (29)$$

For $\vec{\omega} = 0$, this is μ_{12} . To find the third moment, we take another partial derivative, resulting in

$$\begin{aligned} & (-j)^3 \frac{\partial^3}{\partial \omega_3 \partial \omega_2 \partial \omega_1} \Phi(\vec{\omega}) = \\ & (-j)^3 \Phi(\vec{\omega}) [\vec{m}_1^T \vec{\omega} \mu_{23} + \vec{m}_2^T \vec{\omega} \mu_{13} + \vec{m}_3^T \vec{\omega} \mu_{12} - \vec{m}_1^T \vec{\omega} \vec{m}_2^T \vec{\omega} \vec{m}_3^T \vec{\omega}] \end{aligned} \quad (30)$$

The fourth moment is

$$\begin{aligned} & \frac{\partial^4}{\partial \omega_1 \partial \omega_2 \partial \omega_3 \partial \omega_4} \Phi(\vec{\omega}) = \\ & \Phi(\vec{\omega}) [\mu_{12} \mu_{34} + \mu_{13} \mu_{24} + \mu_{14} \mu_{23}] \end{aligned} \quad (31)$$

plus terms which disappear when $\vec{\omega} = 0$. This gives the desired result.