
CONTENTS

Chapter 1	2
Problem 1.1	2
Problem 1.2	3
Problem 1.3	4
Problem 1.4	5
Problem 1.5	6
Problem 1.6	7
Problem 1.7	8
Problem 1.8	9
Problem 1.9	10
Problem 1.10	12
Problem 1.11	13
Problem 1.12	14
Problem 1.13	15
Problem 1.14	16
Problem 1.15	17
Problem 1.16	19
Problem 1.17	20
Problem 1.18	21
Chapter 2	23
Problem 2.1	23
Problem 2.2	24
Problem 2.3	26
Problem 2.4	27
Problem 2.5	28
Problem 2.6	29
Chapter 3	31
Problem 3.1	31
Problem 3.2	32
Problem 3.3	33
Problem 3.4	34
Problem 3.5	35
Problem 3.6	37
Problem 3.7	38
Problem 3.8	39
Problem 3.9	40
Problem 3.10	41
Problem 3.11	42
Problem 3.12	43
Problem 3.13	44
Problem 3.14	45
Problem 3.15	46

Problem 3.16	47
Problem 3.17	48
Problem 3.18	49
Problem 3.19	50
Problem 3.20	51
Problem 3.21	52
Problem 3.22	53
Problem 3.23	54
Problem 3.24	56
Problem 3.25	57
Problem 3.26	58
Problem 3.27	59
Problem 3.28	60
Problem 3.29	61
Problem 3.30	62
Problem 3.31	63
Problem 3.32	64
Problem 3.33	65
Problem 3.34	67
Problem 3.35	68
Problem 3.36	69
Problem 3.37	71
Chapter 4	72
Problem 4.1	72
Problem 4.2	74
Problem 4.3	75
Problem 4.4	76
Problem 4.5	78
Problem 4.6	79
Problem 4.7	81
Problem 4.8	82
Problem 4.9	83
Problem 4.10	84
Problem 4.11	86
Problem 4.12	87
Problem 4.13	88
Problem 4.14	89
Problem 4.15	90
Problem 4.16	91
Problem 4.17	92
Problem 4.18	93
Problem 4.19	94
Problem 4.20	96
Problem 4.21	97
Problem 4.22	98
Problem 4.23	99
Problem 4.24	100
Problem 4.25	101
Problem 4.26	102
Problem 4.27	104
Problem 4.28	108
Problem 4.29	109

Chapter 5	110
Problem 5.1	110
Problem 5.2	111
Problem 5.3	113
Problem 5.4	115
Problem 5.5	116
Problem 5.6	117
Problem 5.7	118
Problem 5.8	120
Problem 5.9	121
Problem 5.10	122
Problem 5.11	123
Problem 5.12	125
Problem 5.13	126
Problem 5.14	127
Problem 5.15	128
Problem 5.16	129
Problem 5.17	132
Problem 5.18	134
Problem 5.19	136
Problem 5.20	138
Problem 5.21	139
Problem 5.22	140
Problem 5.23	141
Problem 5.24	143
Problem 5.25	144
Problem 5.26	145
Problem 5.27	146
Problem 5.28	149
Problem 5.29	151
Problem 5.30	152
Problem 5.31	153
Problem 5.32	155
Problem 5.33	157
Problem 5.34	160
Problem 5.35	168
Problem 5.36	169
Problem 5.37	170
Problem 5.38	173
Problem 5.39	174
Problem 5.40	175
Problem 5.41	177
Problem 5.42	178
Problem 5.43	182
Chapter 6	183
Problem 6.1	183
Problem 6.2	184
Problem 6.3	186
Problem 6.4	188
Problem 6.5	190
Problem 6.6	194
Problem 6.7	196
Problem 6.8	197

Problem 6.9	198
Problem 6.10	199
Problem 6.11	200
Problem 6.12	202
Problem 6.13	203
Problem 6.14	204
Problem 6.15	206
Problem 6.16	207
Problem 6.17	208
Problem 6.18	209
Problem 6.20	211
Problem 6.21	213
Problem 6.22	214
Problem 6.24	216
Problem 6.25	219
Problem 6.29	222
Problem 6.30	223
Problem 6.31	225
Problem 6.32	227
Problem 6.33	229
Problem 6.35	231
Problem 6.37	234
Problem 6.38	235
Problem 6.39	239
Problem 6.40	242
Problem 6.41	245

Chapter 7 **249**

Problem 7.1	249
Problem 7.2	250
Problem 7.3	251
Problem 7.4	253
Problem 7.5	254
Problem 7.6	255
Problem 7.7	256
Problem 7.8	257
Problem 7.9	258
Problem 7.10	260
Problem 7.11	263
Problem 7.15	264
Problem 7.16	269

Chapter 8 **274**

Problem 8.1	274
Problem 8.2	276
Problem 8.3	277
Problem 8.4	278
Problem 8.5	279
Problem 8.6	280
Problem 8.7	283
Problem 8.8	287

Chapter 9	292
Problem 9.1	292
Problem 9.2	294
Problem 9.3	296
Problem 9.4	298
Problem 9.5	299
Problem 9.6	301
Problem 9.7	302
Problem 9.8	304
Problem 9.9	305
Problem 9.10	306
Problem 9.11	307
Problem 9.12	309
Problem 9.13	311
Problem 9.14	313
Problem 9.15	314
Chapter 10	315
Problem 10.1	315
Problem 10.2	316
Problem 10.3	317
Problem 10.4	318
Problem 10.5	319
Problem 10.6	320
Problem 10.7	321
Problem 10.8	322
Problem 10.9	323
Problem 10.10	324
Problem 10.11	325
Problem 10.12	326
Problem 10.13	327
Problem 10.14	328
Problem 10.15	329
Problem 10.16	330
Problem 10.17	331
Problem 10.18	332
Problem 10.19	333
Problem 10.20	335
Problem 10.21	337
Problem 10.22	338
Problem 10.23	339
Problem 10.24	341
Problem 10.25	342
Problem 10.26	344
Problem 10.27	345
Problem 10.28	346
Problem 10.29	347
Problem 10.30	348
Problem 10.31	349
Problem 10.32	351
Problem 10.33	352
Problem 10.34	353
Problem 10.35	355
Problem 10.36	356

Problem 10.38	357
Problem 10.39	358
Problem 10.40	359
Problem 10.41	360
Problem 10.42	362
Problem 10.43	365
Problem 10.47	367
Problem 10.48	368
Chapter 11	370
Problem 11.1	370
Problem 11.2	371
Problem 11.3	372
Problem 11.4	373
Problem 11.5	374
Problem 11.6	375
Problem 11.7	376
Problem 11.8	377
Problem 11.9	378
Problem 11.10	379
Problem 11.11	380
Problem 11.13	381
Problem 11.14	382
Problem 11.15	383
Problem 11.16	384
Problem 11.17	385
Problem 11.18	386
Problem 11.19	387
Problem 11.20	389
Problem 11.21	390
Chapter 13	391
Problem 13.1	391
Problem 13.2	392
Problem 13.3	393
Problem 13.4	395
Problem 13.5	397
Problem 13.6	398
Problem 13.7	399
Problem 13.8	400
Problem 13.9	401
Problem 13.10	402
Problem 13.11	405
Problem 13.12	406
Problem 13.13	407
Problem 13.14	408
Problem 13.15	410
Chapter 14	411
Problem 14.1	411
Problem 14.2	412
Problem 14.3	413
Problem 14.4	414
Problem 14.5	416

Problem 14.6	417
Problem 14.7	418
Problem 14.8	420
Problem 14.9	421
Problem 14.10	422
Problem 14.11	424
Problem 14.12	426
Problem 14.13	428
Problem 14.14	430
Problem 14.15	431
Problem 14.16	433
Problem 14.18	435
Problem 14.19	437
Chapter 15	439
Problem 15.1	439
Problem 15.2	441
Problem 15.3	442
Problem 15.4	443
Problem 15.5	445
Problem 15.6	447
Problem 15.7	448
Problem 15.8	450
Problem 15.9	452
Problem 15.10	454
Problem 15.11	455
Problem 15.12	456
Problem 15.13	458
Problem 15.14	459
Problem 15.15	461
Problem 15.18	463
Chapter 16	464
Problem 16.1	464
Problem 16.2	465
Problem 16.3	466
Problem 16.4	467
Problem 16.5	469
Problem 16.6	471
Problem 16.7	473
Problem 16.8	475
Problem 16.9	477
Problem 16.10	479
Chapter 18	481
Problem 18.1	481
Problem 18.2	483
Problem 18.3	484
Problem 18.4	485
Problem 18.5	486
Problem 18.6	487
Problem 18.7	488

Chapter 19	490
Problem 19.1	490
Problem 19.2	491
Problem 19.3	492
Problem 19.4	493
Problem 19.5	495
Problem 19.7	500
Problem 19.8	502
Problem 19.9	504
Problem 19.10	506
Problem 19.11	507
Problem 19.16	509
Problem 19.21	511
Chapter 20	514
Problem 20.1	514
Problem 20.8	516
Chapter 21	519
Problem 21.1	519
Problem 21.2	522
Problem 21.3	524
Problem 21.4	525
Problem 21.5	526
Problem 21.6	527
Problem 21.7	528
Problem 21.8	529
Problem 21.9	530
Problem 21.10	531
Problem 21.11	535
Problem 21.12	539
Problem 21.13	541
Problem 21.14	542

CHAPTER 1

- 1.1 Solar energy impinging on the outer layer of earth's atmosphere (usually called "solar constant") has been measured as 1367 W/m^2 . What is the solar constant on Mars? (Distance earth to sun = $1.496 \times 10^{11} \text{ m}$, Mars to sun = $2.28 \times 10^{11} \text{ m}$).

Solution

The total energy emitted from the sun $Q = 4\pi R_S^2 \sigma T_{\text{sun}}^4$ goes equally into all directions, so that

$$q_{\text{sc}}(R) = Q/4\pi R^2 = \sigma T_{\text{sun}}^4 \left(\frac{R_S}{R} \right)^2$$
$$\frac{q_{\text{sc mars}}}{q_{\text{sc earth}}} = \left(\frac{R_S}{R_M} \right)^2 \bigg/ \left(\frac{R_S}{R_E} \right)^2 = \left(\frac{R_E}{R_M} \right)^2 = \left(\frac{1.496 \times 10^{11}}{2.28 \times 10^{11}} \right)^2 = 0.4305$$

$$q_{\text{sc mars}} = 0.4305 \times 1367 = 588 \text{ W/m}^2$$

- 1.2 Assuming Earth to be a blackbody, what would be its average temperature if there was no internal heating from the core of earth?

Solution

Without internal heating an energy balance for earth gives

$$\begin{aligned} Q_{\text{absorbed}} &= Q_{\text{emitted}} \\ Q_{\text{abs}} &= q_{\text{sol}} \times A_{\text{proj}} = q_{\text{sol}} \pi R_E^2 \\ Q_{\text{em}} &= \sigma T_E^4 A = 4\pi R_E^2 \sigma T_E^4 \end{aligned}$$

Thus

$$T_E = \left(\frac{q_{\text{sol}}}{4\sigma} \right)^{1/4} = \left(\frac{1367 \text{ W/m}^2}{4 \times 5.670 \times 10^{-8} \text{ W/m}^2 \text{K}^4} \right)^{1/4}$$

$$\boxed{T_E = 279 \text{ K} = 6^\circ \text{C}}$$

- 1.3 Assuming Earth to be a black sphere with a surface temperature of 300 K, what must earth's internal heat generation be in order to maintain that temperature (neglect radiation from the stars, but not the sun) (radius of the earth $R_E = 6.37 \times 10^6$ m).

Solution

Performing an energy balance on earth:

$$\begin{aligned}
 \dot{Q} &= Q_{\text{emitted}} - Q_{\text{absorbed}} = \sigma T_E^4 A - q_{\text{sol}} A_{\text{proj}} \\
 &= 4\pi R_E^2 \sigma T_E^4 - q_{\text{sol}} \pi R_E^2 = \pi R_E^2 (4\sigma T_E^4 - q_{\text{sol}}) \\
 &= \pi \times (6.37 \times 10^6 \text{ m})^2 \left[4 \times 5.670 \times 10^{-8} \frac{\text{W}}{\text{m}^2 \text{K}^4} \times 300^4 \text{ K}^4 - 1367 \frac{\text{W}}{\text{m}^2} \right]
 \end{aligned}$$

$$\boxed{\dot{Q} = 6.00 \times 10^{16} \text{ W}}$$

- 1.4** To estimate the diameter of the sun, one may use solar radiation data. The solar energy impinging onto the earth's atmosphere (called the "solar constant") has been measured as 1367 W/m^2 . Assuming that the sun may be approximated to have a black surface with an effective temperature of 5777 K , estimate the diameter of the sun (distance sun to earth $S_{ES} \approx 1.496 \times 10^{11} \text{ m}$).

Solution

The sun's energy travels equally into all directions. Thus, on a spherical "shell" around the sun (such that earth's orbit lies on that shell): total energy leaving sun

$$Q_{\text{sun}} = q_{\text{sol}} \times 4\pi S_{ES}^2 = \sigma T_{\text{sun}}^4 4\pi R_S^2.$$

Thus

$$\begin{aligned} R_S &= S_{ES} \left(\frac{q_{\text{sol}}}{\sigma T_{\text{sun}}^4} \right)^{1/2} \\ &= 1.496 \times 10^{11} \text{ m} \left(\frac{1367 \text{ W/m}^2}{5.670 \times 10^{-8} \times 5777^4 \text{ W/m}^2} \right)^{1/2} \end{aligned}$$

$$\boxed{R_S = 6.96 \times 10^8 \text{ m}}$$

- 1.5 Solar energy impinging on the outer layer of earth's atmosphere (usually called "solar constant") has been measured as 1367 W/m^2 . Assuming the sun may be approximated as having a surface that behaves like a blackbody, estimate its effective surface temperature. (Distance sun to earth $S_{ES} \approx 1.496 \times 10^{11} \text{ m}$, radius of sun $R_S \approx 6.96 \times 10^8 \text{ m}$).

Solution

The sun's energy travels equally into all directions. Thus, on a spherical "shell" around the sun (such that the earth's orbit lies on that shell): total energy leaving sun

$$Q_{\text{sun}} = q_{\text{sol}} \times 4\pi S_{ES}^2 = \sigma T_{\text{sun}}^2 4\pi R_S^2.$$

Thus

$$\begin{aligned} T_{\text{sun}} &= \left(\frac{q_{\text{sol}}}{\sigma} \right)^{1/4} \left(\frac{S_{ES}}{R_S} \right)^{1/2} \\ &= \left(\frac{1367 \text{ W/m}^2}{5.670 \times 10^{-8} \text{ W/m}^2 \text{K}^4} \right)^{1/4} \left(\frac{1.496 \times 10^{11}}{6.96 \times 10^8} \right)^{1/2} \end{aligned}$$

$T_{\text{sun}} = 5777 \text{ K}$

- 1.6** A rocket in space may be approximated as a black cylinder of length $L = 20$ m and diameter $D = 2$ m. It flies past the sun at a distance of 140 million km such that the cylinder axis is perpendicular to the sun's rays. Assuming that (i) the sun is a blackbody at 5777 K and (ii) the cylinder has a high conductivity (i.e., is essentially isothermal), what is the temperature of the rocket? (Radius of sun $R_S = 696,000$ km; neglect radiation from earth and the stars).

Solution

The total energy emitted from the sun $Q = 4\pi R_S^2 \sigma T_{\text{sun}}^4$ goes equally into all directions, so that the flux arriving at the rocket a distance S_{RS} away from the sun is

$$q_R = Q/4\pi S_{RS}^2 = \sigma T_{\text{sun}}^4 \left(\frac{R_S}{S_{RS}} \right)^2 = \text{flux/unit projected area of rocket}$$

$$\begin{aligned} \text{Energy emitted} &= A_R \sigma T_R^4 = (2\pi R^2 + 2\pi RL) \sigma T_R^4 = 2\pi R(R + L) \sigma T_R^4 \\ &= \text{Energy absorbed} = A_p q_R = 2RL \sigma T_{\text{sun}}^4 \left(\frac{R_S}{S_{RS}} \right)^2 \end{aligned}$$

$$T_R^4 = \frac{2RL}{2\pi R(R + L)} T_{\text{sun}}^4 \left(\frac{R_S}{S_{RS}} \right)^2 = \frac{1}{\pi} \frac{20}{1 + 20} (5777 \text{ K})^4 \left(\frac{6.96 \times 10^5}{140 \times 10^6} \right)^2$$

$$T_R = 302 \text{ K.}$$

- 1.7 A black sphere of very high conductivity (i.e., isothermal) is orbiting earth. What is its temperature? (Consider the sun but neglect radiation from the earth and the stars). What would be the temperature of the sphere if it were coated with a material that behaves like a black body for wavelengths between $0.4\ \mu\text{m}$ and $3\ \mu\text{m}$, but does not absorb and emit at other wavelengths?

Solution

Making an energy balance on the sphere

(a)

$$\begin{aligned} Q_{\text{absorbed}} &= Q_{\text{emitted}} \\ Q_{\text{absorbed}} &= q_{\text{sol}} A_{\text{proj}} = q_{\text{sol}} \pi R^2 \\ Q_{\text{emitted}} &= \sigma T^4 A = \sigma T^4 4\pi R^2 \end{aligned}$$

Thus

$$T = \left(\frac{q_{\text{sol}}}{4\sigma} \right)^{1/4} = \left(\frac{1367\ \text{W/m}^2}{4 \times 5.670 \times 10^{-8}\ \text{W/m}^2\text{K}^4} \right)^{1/4}$$

$$\boxed{T = 279\ \text{K} = 6^\circ\text{C}}$$

(b) If absorption and emission takes place only over the wavelength interval $0.4\ \mu\text{m} < \lambda < 3\ \mu\text{m}$, both Q_{abs} and Q_{em} will be reduced. Since the sphere is spectrally black, we have

$$Q_{\text{em}} = 4\pi R^2 \int_{0.4\ \mu\text{m}}^{3\ \mu\text{m}} E_{b\lambda} d\lambda = 4\pi R^2 \sigma T^4 [f(3\ \mu\text{m} \times T) - f(0.4\ \mu\text{m} \times T)]$$

Solar emission is black, i.e., $q_{\text{sol}} \propto E_{b\lambda}(T_{\text{sun}})$, or

$$\frac{q_{\text{sol},\lambda}}{q_{\text{sol}}} = \frac{E_{b\lambda}(T_{\text{sun}})}{\sigma T_{\text{sun}}^4},$$

and

$$\begin{aligned} Q_{\text{abs}} &= \pi R^2 \int_{0.4\ \mu\text{m}}^{3\ \mu\text{m}} q_{\text{sol},\lambda} d\lambda = \pi R^2 \frac{q_{\text{sol}}}{\sigma T_{\text{sun}}^4} \int_{0.4\ \mu\text{m}}^{3\ \mu\text{m}} E_{b\lambda}(T_{\text{sun}}) d\lambda \\ &= \pi R^2 q_{\text{sol}} [f(3\ \mu\text{m} \times T_{\text{sun}}) - f(0.4\ \mu\text{m} \times T_{\text{sun}})] \end{aligned}$$

Thus,

$$\begin{aligned} 4\sigma T^4 [f(3\ \mu\text{m} T) - f(0.4\ \mu\text{m} T)] &= q_{\text{sol}} [f(3\ \mu\text{m} T_{\text{sun}}) - f(0.4\ \mu\text{m} T_{\text{sun}})] \\ &= 1367\ \text{W/m}^2 [f(17,331) - f(2311)] \end{aligned}$$

$$\begin{aligned} T^4 [f(3\ \mu\text{m} T) - f(0.4\ \mu\text{m} T)] &= \frac{1367}{4 \times 5.670 \times 10^{-8}} (0.9788 - 0.1222) \\ &= 5.1630 \times 10^9\ \text{K}^4 \end{aligned}$$

This nonlinear relation must be solved by iteration, leading to $T \approx 600\ \text{K}$ ($< 601\ \text{K}$)

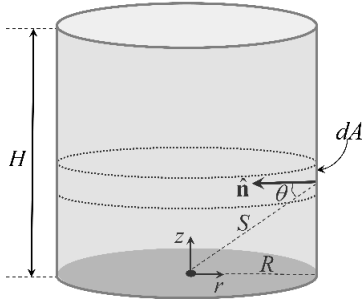
$$600^4 \times [0.03934 - 0] = 5.0985 \times 10^9$$

$$\boxed{T = 600\ \text{K}}$$

- 1.8 Derive an expression for the solid angle subtended by the curved surface of a cylinder of height H and radius R on the center of its base. If the cylinder is very tall compared to its radius, what do you expect this solid angle to be? Verify your expectation mathematically using the expression you have derived.

Solution

We select a cylindrical coordinate system with the origin placed at the center of the cylinder's base, as shown in the figure below.



From the figure, $\cos \theta = R / S$. Next, using the definition of the solid angle, and noting that $dA = 2\pi R dz$, we obtain

$$\Omega = \int_A \frac{\cos \theta dA}{S^2} = \int_0^H \frac{R}{S^3} 2\pi R dz = 2\pi R^2 \int_0^H \frac{1}{S^3} dz$$

Finally, noting that $S = \sqrt{R^2 + z^2}$, we obtain

$$\Omega = 2\pi R^2 \int_0^H \frac{1}{S^3} dz = 2\pi R^2 \int_0^H \frac{dz}{[R^2 + z^2]^{3/2}} = 2\pi R^2 \frac{1}{R^2} \left[\frac{z}{\sqrt{R^2 + z^2}} \right]_0^H = \frac{2\pi H}{\sqrt{R^2 + H^2}}$$

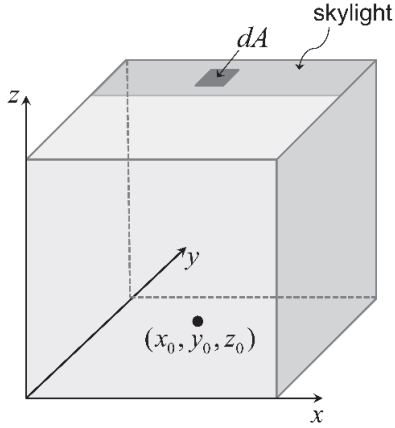
If the cylinder is very long compared to its radius, then the solid angle subtended by its upper flat surface will be negligible. Therefore, the solid angle subtended by the curved surface of the cylinder will be very close to 2π , almost as if the base is completely enclosed by the curved surface. Mathematically, the expression derived above for the solid angle may be re-written as

$$\Omega = \frac{2\pi H}{\sqrt{R^2 + H^2}} = \frac{2\pi}{\sqrt{(R/H)^2 + 1}}$$

In the limit $H \gg R$ (or $R/H \ll 1$), this expression reduces to 2π , thereby confirming our physical intuition.

- 1.9 An interior space of a building, with dimensions L_x , L_y , and L_z , has a skylight on the roof, as shown in the figure. The skylight is of width $L_y/4$. In order to properly design the interior lighting, an architect is interested in knowing the solid angle subtended on any point, (x_0, y_0, z_0) , inside this space by the skylight. Derive an expression for this solid angle.

Solution



Since the skylight is on the x - y plane, it follows that $dA = dx dy$. Let the target point (on which the subtended solid angle is to be derived) have coordinates (x_0, y_0, z_0) , as shown in the figure.

The vector $\mathbf{s}$ connects the differential area dA to this point. Therefore,

$S = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (L_z - z_0)^2}$, and $\cos \theta = (L_z - z_0) / S$, where θ is the acute angle between the vector $\mathbf{s}$ and the downward pointing surface normal from the skylight surface.

Substituting these expressions into the definition of the solid angle, we obtain

$$\Omega = \int_{A_{\text{skylight}}} \frac{\cos \theta dA}{S^2} = \int_{A_{\text{skylight}}} \frac{(L_z - z_0) dA}{S^3} = \int_{y=3L_y/4}^{L_y} \int_{x=0}^{L_x} \frac{(L_z - z_0) dx dy}{[(x - x_0)^2 + (y - y_0)^2 + (L_z - z_0)^2]^{3/2}}$$

Performing the integration over x first, we obtain

$$\begin{aligned}
\Omega &= (L_z - z_0) \int_{y=3L_y/4}^{L_y} \left[\frac{(x - x_0)}{[(y - y_0)^2 + (L_z - z_0)^2][(x - x_0)^2 + (y - y_0)^2 + (L_z - z_0)^2]^{1/2}} \right]_0^{L_x} dy \\
&= (L_z - z_0)(L_x - x_0) \int_{y=3L_y/4}^{L_y} \frac{1}{[(y - y_0)^2 + (L_z - z_0)^2][(y - y_0)^2 + (L_x - x_0)^2 + (L_z - z_0)^2]^{1/2}} dy \\
&\quad + (L_z - z_0)x_0 \int_{y=3L_y/4}^{L_y} \frac{1}{[(y - y_0)^2 + (L_z - z_0)^2][(y - y_0)^2 + (x_0)^2 + (L_z - z_0)^2]^{1/2}} dy
\end{aligned}$$

Next, performing the integration over y , we obtain

$$\Omega = (L_z - z_0)(L_x - x_0)I_1 + (L_z - z_0)x_0I_2$$

where

$$I_1 = \int_{y=3L_y/4}^{L_y} \frac{1}{[(y - y_0)^2 + (L_z - z_0)^2][(y - y_0)^2 + (L_x - x_0)^2 + (L_z - z_0)^2]^{1/2}} dy, \text{ and}$$

$$I_2 = \int_{y=3L_y/4}^{L_y} \frac{1}{[(y - y_0)^2 + (L_z - z_0)^2][(y - y_0)^2 + (x_0)^2 + (L_z - z_0)^2]^{1/2}} dy$$

Both integrals can be written in the form $I = \int \frac{1}{[(y - y_0)^2 + b^2][(y - y_0)^2 + a^2]^{1/2}} dy$

The result of this integral may be written in three parts, as follows:

$$I = \begin{cases} \frac{1}{2b\sqrt{b^2 - a^2}} \ln \left[\frac{(y - y_0)\sqrt{b^2 - a^2} + b\sqrt{(y - y_0)^2 + a^2}}{(y - y_0)\sqrt{b^2 - a^2} - b\sqrt{(y - y_0)^2 + a^2}} \right] & \text{if } b^2 > a^2 \\ \frac{1}{b\sqrt{a^2 - b^2}} \tan^{-1} \left[\frac{(y - y_0)\sqrt{a^2 - b^2}}{b\sqrt{(y - y_0)^2 + a^2}} \right] & \text{if } a^2 > b^2 \\ \frac{(y - y_0)}{a^2\sqrt{(y - y_0)^2 + a^2}} & \text{if } b^2 = a^2 \end{cases}$$

The appropriate expression should be used depending on the values of (x_0, y_0, z_0) and (L_x, L_y, L_z) .

- 1.10** A 100 Watt light bulb may be considered to be an isothermal black sphere at a certain temperature. If the light flux (i.e., visible light, $0.4 \mu\text{m} < \lambda < 0.7 \mu\text{m}$) impinging on the floor directly (2.5 m) below the bulb is 42.6 mW/m^2 , what is the light bulb's effective temperature? What is its efficiency?

Solution

The total heat rate (100 W) leaving the light bulb will go at equal per-unit-area amounts through any (hypothetical) sphere around the bulb. Take a sphere which has the spot on the floor on its surface (radius $h = 2.5 \text{ m}$)

$$q_{\text{floor,visible}} = q_{\text{fv}} = [f(\lambda_2 T) - f(\lambda_1 T)] \frac{Q_{\text{bulb}}}{4\pi h^2}$$

$$f(\lambda_2 T) - f(\lambda_1 T) = 4\pi h^2 q_{\text{fv}} / Q_{\text{bulb}} = 4\pi (2.5)^2 \text{m}^2 (42.6 \times 10^{-3} \frac{\text{W}}{\text{m}^2}) / 100 \text{ W}$$

$$= 0.03346$$

By trial and error:

$$T = 2500 \text{ K} \rightarrow \lambda_1 T = 0.4 \mu\text{m} \times 2500 \text{ K} = 1000 \mu\text{m K}$$

$$\lambda_2 T = 0.7 \mu\text{m} \times 2500 \text{ K} = 1750 \mu\text{m K}$$

$$f(\lambda_2 T) - f(\lambda_1 T) = 0.03369 - 0.00032 = 0.03337$$

$T = 2500 \text{ K}; \quad \eta = 3.34\% \quad \text{is converted into visible light}$
--

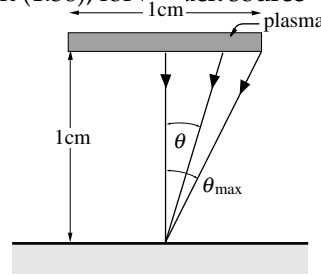
The answers for $f(\lambda_1 T)$ and $f(\lambda_2 T)$ are the same whether Appendix C or program planck.exe are employed, since no interpolation is necessary.

- 1.11** When a metallic surface is irradiated with a highly concentrated laser beam, a plume of plasma (i.e., a gas consisting of ions and free electrons) is formed above the surface that absorbs the laser's energy, often blocking it from reaching the surface. Assume that a plasma of 1 cm diameter is located 1 cm above the surface, and that the plasma behaves like a blackbody at 20,000 K. Based on these assumptions calculate the radiative heat flux and the total radiation pressure on the metal directly under the center of the plasma.

Solution

Radiation emitted from the plasma ($\approx$ black disk) hits the spot below it generating an incoming radiative heat flux and a radiative pressure. From equation (1.38), for a black source

$$\begin{aligned}
 q_{\text{in}} &= \int_{2\pi} I_b \cos \theta \, d\Omega \\
 &= I_b \int_{\Omega_{\text{plasma}}} \cos \theta \, d\Omega \\
 &= 2\pi I_b \int_0^{\theta_{\text{max}}} \cos \theta \sin \theta \, d\theta \\
 &= E_b \sin^2 \theta_{\text{max}} = E_b \times \frac{1}{5} = \frac{5.670 \times 10^{-8} \times 20,000^4 \, \text{W/m}^2}{5}
 \end{aligned}$$



$$q_{\text{in}} = 1.814 \times 10^9 \, \text{W/m}^2 = 181.4 \, \text{kW/cm}^2$$

This should be compared to the maximum flux at the center of an unobstructed laser beam, which may be several MW/cm².

The radiation pressure is found similarly, from equation (1.45) as

$$\begin{aligned}
 p &= \frac{2\pi I_b}{c} \int_0^{\theta_{\text{max}}} \cos^2 \theta \sin \theta \, d\theta = \frac{2E_b}{3c} (1 - \cos^2 \theta_{\text{max}}) = \frac{2E_b}{3c} \left[1 - \left(\frac{4}{5} \right)^{3/2} \right] \\
 &= \frac{2}{3} \frac{5.670 \times 10^{-8} \times 20,000^4 \, \text{W/m}^2}{3 \times 10^8 \, \text{m/s}} [1 - 0.7155] = 5.735 \frac{\text{Ws}}{\text{m}^3} = 5.735 \frac{\text{N}}{\text{m}^2}
 \end{aligned}$$

$$p = 5.735 \, \text{N/m}^2$$

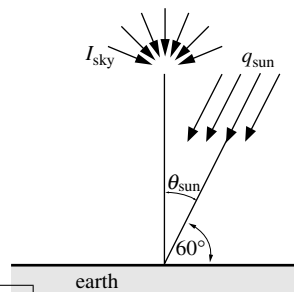
- 1.12** Solar energy incident on the surface of the Earth may be broken into two parts: A direct component (traveling unimpeded through the atmosphere) and a sky component (reaching the surface after being scattered by the atmosphere). On a clear day the direct solar heat flux has been determined as $q_{\text{sun}} = 1000 \text{ W/m}^2$ (per unit area normal to the rays), while the intensity of the sky component has been found to be diffuse (i.e., the intensity of the sky radiation hitting the surface is the same for all directions) and $I_{\text{sky}} = 70 \text{ W/m}^2\text{sr}$. Determine the total solar irradiation onto earth's surface if the sun is located 60° above the horizon (i.e., 30° from the normal).

Solution

Since earth's surface is tilted away from the sun, less energy per unit surface area hits earth than is carried by the sunshine (per unit area normal to the rays), as seen from Fig. 1-8 (by a factor of $\cos \theta_{\text{sun}} = \cos 30^\circ$). Thus from equation (1.38)

$$\begin{aligned}
 q_{\text{in}} &= q_{\text{sun}} \cos \theta_{\text{sun}} + \int_{2\pi} I_{\text{sky}} \cos \theta \, d\Omega \\
 &= q_{\text{sun}} \cos \theta_{\text{sun}} + \int_0^{2\pi} \int_0^{\pi/2} \cos \theta \sin \theta \, d\theta \, d\psi \\
 &= q_{\text{sun}} \cos \theta_{\text{sun}} + \pi I_{\text{sky}} \\
 &= 1000 \times \cos 30^\circ + \pi \times 70 = 1086 \text{ W/m}^2
 \end{aligned}$$

$q_{\text{in}} = 1086 \text{ W/m}^2$



- 1.13** If the average absorptance of the Earth's surface is denoted by α_e , and its average emittance is denoted by ϵ_e , derive an expression for the equilibrium surface temperature of the Earth, T_e , in terms of these quantities and the solar constant, q_{sol} . Assume that the atmosphere is absent and there is negligible heat generation within the Earth's core. Using the value of the solar constant, given by equation (1.43), $\alpha_e = 0.7$, and $\epsilon_e = 1.0$, estimate the Earth's equilibrium surface temperature. Comment on your result. Would the result change if the Earth's surface was gray? Would the Earth's surface temperature change if its radius was five times larger?

Solution

Although the radiative energy balance, locally, of any point on the Earth's surface is transient by nature, averaged over a day or several days, half the Earth's surface always sees the sun (incident energy from the sun), and the whole surface always loses energy by emission regardless of whether it is facing the sun or not. The net energy per unit time absorbed by the Earth follows from equation (1.44):

$$Q_{abs} = -\alpha_e (\pi R_E^2) \int_0^\infty H_{s\lambda} d\lambda = -\alpha_e (\pi R_E^2) q_{sol},$$

where R_E is the radius of the Earth. The area, πR_E^2 , is the projected area of the Earth facing the sun. A formal derivation of why the projected area is relevant rather than the total area is presented in Example 3.8. The net energy emitted by the entire surface of the Earth, depends on its own temperature, and is written as

$$Q_{em} = \epsilon_e (4\pi R_E^2) \int_0^\infty E_{b\lambda}(T_e) d\lambda = \epsilon_e (4\pi R_E^2) \sigma T_e^4.$$

Assuming that conduction from the hot core of the Earth is negligible, its surface temperature will reach an equilibrium (steady state) if the absorbed energy balances the emitted energy. Thus, we get

$$\sigma T_e^4 = \frac{1}{4} \frac{\alpha_e}{\epsilon_e} q_{sol},$$

solution of which yields $T_e = 254.81$ K. The computed equilibrium surface temperature of T_e is considerably lower than the recorded average temperature of the Earth's surface, which is about 14.9°C (as of 2019), due to the fact that the role of the atmosphere has been ignored in the present estimate.

If the Earth's surface is assumed to be gray, then $\alpha_e = \epsilon_e$ by Kirchhoff's law, and the Earth's temperature would become independent of its surface radiative properties according to the above equation, which would defy any physical intuition.

Increasing the Earth's radius would have no impact on the estimated temperature, since the radius does not appear in the above equation.

- 1.14** A window (consisting of a vertical sheet of glass) is exposed to direct sunshine at a strength of 1000 W/m^2 . The window is pointing due south, while the sun is in the southwest, 30° above the horizon. Estimate the amount of solar energy that (i) penetrates into the building, (ii) is absorbed by the window, and (iii) is reflected by the window. The window is made of (a) plain glass, (b) tinted glass, whose radiative properties may be approximated by

$$\begin{aligned}\rho_\lambda &= 0.08 \quad \text{for all wavelengths (both glasses),} \\ \tau_\lambda &= \begin{cases} 0.90 & \text{for } 0.35 \mu\text{m} < \lambda < 2.7 \mu\text{m} \\ 0 & \text{for all other wavelengths} \end{cases} \quad (\text{plain glass}) \\ \tau_\lambda &= \begin{cases} 0.90 & \text{for } 0.5 \mu\text{m} < \lambda < 1.4 \mu\text{m} \\ 0 & \text{for all other wavelengths} \end{cases} \quad (\text{tinted glass}).\end{aligned}$$

(c) By what fraction is the amount of visible light ($0.4 \mu\text{m} < \lambda < 0.7 \mu\text{m}$) reduced, if tinted rather than plain glass is used? How would you modify this statement in the light of Fig. 1-11?

Solution

If we want to determine how much sunshine penetrates through the window (for the entire window area, or per unit window area), we need to determine the angle between a normal to the window ($\hat{n}$) and a vector pointing to the sun ($\hat{s}$), or $\cos \beta = \hat{n} \cdot \hat{s}$. From geometric relations it follows that $\cos \beta = \cos 30^\circ \times \cos 45^\circ = 0.6124$. Thus sunshine on the window, per unit window area, is $q_{\text{in}} = 612.4 \text{ W/m}^2$. Of that the fraction 0.08 is reflected, i.e.,

$$q_{\text{ref}} = 0.08 \times 612.4 = 49.0 \text{ W/m}^2 \text{ of window area.}$$

(a) Plain glass: The transmitted sunshine may be calculated as

$$\begin{aligned}q_{\text{trans}} &= \int_0^\infty \tau_\lambda q_{\text{in},\lambda} d\lambda = 0.90 \int_{0.35 \mu\text{m}}^{2.7 \mu\text{m}} q_{\text{in},\lambda} d\lambda \\ &= 0.90 q_{\text{in}} [f(2.7 \mu\text{m } T_{\text{sun}}) - f(0.35 \mu\text{m } T_{\text{sun}})] \\ &= 0.90 \times 612.4 \text{ W/m}^2 [f(15,598 \mu\text{m K}) - f(2022 \mu\text{m K})] \\ &= 0.90 \times 612.4 \times (0.9720 - 0.0702) = 497.0 \text{ W/m}^2\end{aligned}$$

It follows that

$$q_{\text{abs}} = q_{\text{in}} - q_{\text{ref}} - q_{\text{trans}} = 612.4 - 49.0 - 497.4 = 66.0 \text{ W/m}^2$$

(b) Tinted glass: Similarly,

$$\begin{aligned}q_{\text{trans}} &= 0.90 \times q_{\text{in}} [f(1.4 \mu\text{m } \times 5777 \text{ K}) - f(0.5 \mu\text{m } \times 5777 \text{ K})] \\ &= 0.90 \times 612.4 \times [f(8088) - f(2889)] \text{ W/m}^2 \\ &= 0.90 \times 612.4 \times (0.8597 - 0.2481) = 337.1 \text{ W/m}^2,\end{aligned}$$

$$q_{\text{trans}} = 337.1 \text{ W/m}^2$$

i.e., heat gain is reduced by 160 W/m^2 (or 32%).

(c) The tinted glass loses some visible light, since it is not transmissive for $0.4 \mu\text{m} < \lambda < 0.5 \mu\text{m}$. The fractional reduction is

$$\begin{aligned}\text{fractional reduction} &= \frac{0.90 q_{\text{in}} [f(0.5 \mu\text{m } T_{\text{sun}}) - f(0.4 \mu\text{m } T_{\text{sun}})]}{0.90 q_{\text{in}} [f(0.7 \mu\text{m } T_{\text{sun}}) - f(0.4 \mu\text{m } T_{\text{sun}})]} \\ &= \frac{f(2889) - f(2311)}{f(4044) - f(2311)} = \frac{0.2481 - 0.1222}{0.4888 - 0.1222}\end{aligned}$$

$$\text{fractional reduction} = 34.3\%$$

Therefore, the tinted glass appears to cause an equivalent loss of visible light. However, Fig. 1-11 shows that the human eye is not very responsive to wavelengths below $\approx 0.5 \mu\text{m}$, so that the actual reduction of visible light is considerably less.

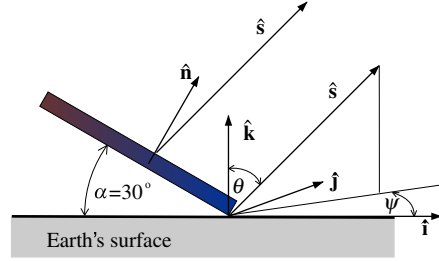
- 1.15** On an overcast day the directional behavior of the intensity of solar radiation reaching the surface of the earth after being scattered by the atmosphere may be approximated as $I_{\text{sky}}(\theta) = I_{\text{sky}}(\theta=0) \cos \theta$, where θ is measured from the surface normal. For a day with $I_{\text{sky}}(0) = 100 \text{ W/m}^2\text{sr}$ determine the solar irradiation hitting a solar collector, if the collector is (a) horizontal, (b) tilted from the horizontal by 30° . Neglect radiation from the earth's surface hitting the collector (by emission or reflection).

Solution

(a) For a horizontal collector the solar irradiation is readily determined from equation (1.38) as

$$\begin{aligned} q_{\text{in}} &= \int_0^{2\pi} \int_0^{\pi/2} (I_0 \cos \theta) \cos \theta \sin \theta d\theta d\psi \\ &= 2\pi I_0 \int_0^{\pi/2} \cos^2 \theta \sin \theta d\theta = \frac{2\pi}{3} I_0 = \frac{2\pi}{3} \times 100 \end{aligned}$$

$$q_{\text{in}} = 209.4 \text{ W/m}^2$$



(b) To evaluate q_{in} for a tilted collector, equation (1.38) becomes

$$q_{\text{in}} = \int_{\hat{\mathbf{n}} \cdot \hat{\mathbf{s}} > 0} I_0 (\hat{\mathbf{k}} \cdot \hat{\mathbf{s}}) (\hat{\mathbf{n}} \cdot \hat{\mathbf{s}}) d\Omega,$$

where $\hat{\mathbf{k}} \cdot \hat{\mathbf{s}} = \cos \theta$ gives the variation of sky intensity with sky polar angle θ , while $\hat{\mathbf{n}} \cdot \hat{\mathbf{s}} = \cos \theta'$ is the cosine of the angle between the surface normal and a unit vector pointing into the sky, $\hat{\mathbf{s}}$. This unit vector may be expressed in terms of $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ (unit vectors for an earth-bound coordinate system), with

$$\hat{\mathbf{i}}' = \cos \alpha \hat{\mathbf{i}} - \sin \alpha \hat{\mathbf{k}}, \quad \hat{\mathbf{j}}' = \hat{\mathbf{j}}, \quad \hat{\mathbf{n}} = \sin \alpha \hat{\mathbf{i}} + \cos \alpha \hat{\mathbf{k}},$$

as

$$\begin{aligned} \hat{\mathbf{s}} &= \sin \theta \cos \psi \hat{\mathbf{i}} + \sin \theta \sin \psi \hat{\mathbf{j}} + \cos \theta \hat{\mathbf{k}} \\ &= \sin \theta' \cos \psi' \hat{\mathbf{i}}' + \sin \theta' \sin \psi' \hat{\mathbf{j}}' + \cos \theta' \hat{\mathbf{n}}. \end{aligned}$$

Substituting for $\hat{\mathbf{i}}', \hat{\mathbf{j}}'$ and $\hat{\mathbf{n}}, \hat{\mathbf{s}}(\theta', \psi')$ becomes

$$\begin{aligned} \hat{\mathbf{s}} &= (\sin \theta' \cos \psi \cos \alpha + \cos \theta' \sin \alpha) \hat{\mathbf{i}} + \sin \theta' \sin \psi \hat{\mathbf{j}} \\ &\quad + (\cos \theta' \cos \alpha - \sin \theta' \cos \psi' \sin \alpha) \hat{\mathbf{k}}. \end{aligned}$$

Equation (1.38) may be evaluated by evaluating $\hat{\mathbf{k}} \cdot \hat{\mathbf{s}}$ in collector coordinates (i.e., integrating over θ' and ψ'), or by determining $\hat{\mathbf{n}} \cdot \hat{\mathbf{s}}$ in global coordinates (integrating over θ and ψ). Choosing the former, one obtains

$$q_{\text{in}} = \int_{\substack{\hat{\mathbf{k}} \cdot \hat{\mathbf{s}} > 0 \\ \hat{\mathbf{n}} \cdot \hat{\mathbf{s}} > 0}} I_0 (\cos \theta' \cos \alpha - \sin \theta' \cos \psi' \sin \alpha) \cos \theta' \sin \theta' d\theta' d\psi'$$

The integration limits imply that only directions above the collector ($0 < \theta < \pi/2$) and above the horizontal

$$\hat{\mathbf{k}} \cdot \hat{\mathbf{s}} = \cos \theta = \cos \theta' \cos \alpha - \sin \theta' \cos \psi' \sin \alpha > 0$$

are to be considered. Solving the last expression for $\cos \psi'$ yields

$$\cos \psi' < \cot \alpha \cot \theta',$$

and using symmetry

$$q_{\text{in}} = 2I_0 \int_0^{\pi/2} \int_{\cos^{-1}(\cot \alpha \cot \theta')}^{\pi} (\cos \theta' \cos \alpha - \sin \theta' \cos \psi' \sin \alpha) \times \cos \theta' \sin \theta' d\psi' d\theta'.$$

Any radiation from earth's surface hits the collector with a polar angle of $\theta' > 60^\circ$, making a negligible contribution to q_{in} . Thus, as a first approximation, the lower limit for ψ' may be replaced by 0, such that

$$\begin{aligned} q_{\text{in}} &= 2I_b \int_0^{\pi/2} \int_0^{\pi} (\cos \theta' \cos \alpha - \sin \theta' \cos \psi' \sin \alpha) \cos \theta' \sin \theta' d\psi' d\theta' \\ &= \frac{2\pi}{3} I_0 \cos \alpha = \frac{2\pi}{3} \times 100 \times \cos 30^\circ = 181.4 \text{ W/m}^2. \end{aligned}$$

$q_{\text{in}} = 181.4 \text{ W/m}^2$

This number is a little bit too low because the additionally considered radiation from the earth has a $\hat{\mathbf{k}} \cdot \hat{\mathbf{s}} = \cos \theta < 0$.

- 1.16** A 100 W light bulb is rated to have a total light output of 1750 lm. Assuming the light bulb to consist of a small, black, radiating body (the light filament) enclosed in a glass envelope (with a transmittance $\tau_g = 0.9$ throughout the visible wavelengths), estimate the filament's temperature. If the filament has an emittance of $\epsilon_f = 0.7$ (constant for all wavelengths and directions), how does it affect its temperature?

Solution

The total heat rate leaving the light bulb by radiation (going equally into all directions) is

$$Q_{\text{bulb}} = \tau Q_{\text{fil}} = \tau (A_{\text{fil}} \epsilon_{\text{fil}} \sigma T_{\text{fil}}^4).$$

Thus the radiative heat flux on a large, hypothetical sphere of radius R surrounding the bulb is

$$q(R) = \frac{Q_{\text{bulb}}}{A_{\text{sphere}}} = \frac{A_{\text{fil}}}{A_{\text{sphere}}} \tau \epsilon_{\text{fil}} \pi I_b(T_{\text{fil}}) = I(R) d\Omega_{\text{fil}}$$

where $I(R)$ indicates that the flux at R is coming from a very small solid angle normal to the surface and is, therefore, simply a normal intensity multiplied by the small solid angle.

From equation (1.50) the luminous flux at R is

$$q_{\text{lum}}(R) = L d\Omega_{\text{fil}} = 286 \frac{\text{lm}}{\text{W}} \int_{0.4 \mu\text{m}}^{0.7 \mu\text{m}} I_{\lambda}(R) d\lambda d\Omega_{\text{fil}}$$

With gray (i.e., constant) values for ϵ_{fil} and τ (at least over the visible wavelengths), this makes $I_{\lambda}(R) \propto I_{b\lambda}(T_{\text{fil}})$, and

$$\begin{aligned} q_{\text{lum}}(R) &= 286 \frac{\text{lm}}{\text{W}} I(R) d\Omega_{\text{fil}} [f(0.7 \mu\text{m } T_{\text{fil}}) - f(0.4 \mu\text{m } T_{\text{fil}})]. \\ Q_{\text{lum}} &= q_{\text{lum}}(R) A_{\text{sphere}} \\ &= 286 \frac{\text{lm}}{\text{W}} A_{\text{fil}} \tau \epsilon_{\text{fil}} \pi I_b(T_{\text{fil}}) [f(0.7 \mu\text{m } T_{\text{fil}}) - f(0.4 \mu\text{m } T_{\text{fil}})] \\ &= 286 \frac{\text{lm}}{\text{W}} \tau Q_{\text{fil}} [f(0.7 \mu\text{m } T_{\text{fil}}) - f(0.4 \mu\text{m } T_{\text{fil}})], \end{aligned}$$

where Q_{lum} is the total luminous flux leaving the bulb. Thus

$$\begin{aligned} f(0.7 T_{\text{fil}}) - f(0.4 T_{\text{fil}}) &= \frac{Q_{\text{lum}}}{286(\text{lm/W}) \tau Q_{\text{fil}}} = \frac{1750 \text{ lm}}{286(\text{lm/W}) \times 0.9 \times 100 \text{ W}} \\ &= 0.06799. \end{aligned}$$

By trial and error, one finds from Appendix C, for $T_{\text{fil}} \approx 2880 \text{ K}$

$$f(0.7 \times 2880) - f(0.4 \times 2880) = 0.06923 - 0.00145 = 0.06778.$$

$T_{\text{fil}} \approx 2880 \text{ K}$

This expression does not include the filament emittance: $\epsilon_f = 0.7$ would leave T_{fil} unchanged (however, A_{fil} would have to increase by a factor of $1/0.7 = 1.43$, in order to emit 100 W of power).

Alternatively, we could follow Example 1.7, i.e., use

$$q_{\text{lum}}(R) = 286 \frac{\text{lm}}{\text{W}} \int_{0.4 \mu\text{m}}^{0.7 \mu\text{m}} q_{\lambda}(R) d\lambda,$$

which is valid for arbitrary incoming flux. Multiplying by the area of our hypothetical sphere A_{sphere} , we obtain

$$Q_{\text{lum}}(R) = 286 \frac{\text{lm}}{\text{W}} \int_{0.4 \mu\text{m}}^{0.7 \mu\text{m}} Q_{\lambda}(R) d\lambda = 286 \frac{\text{lm}}{\text{W}} \tau Q_{\text{fil}} [f(0.7 \mu\text{m } T_{\text{fil}}) - f(0.4 \mu\text{m } T_{\text{fil}})],$$

as before.

- 1.17 A *pyrometer* is a device with which the temperature of a surface may be determined remotely by measuring the radiative energy falling onto a detector. Consider a black detector of $1\text{ mm} \times 1\text{ mm}$ area, that is exposed to a 1 cm^2 hole in a furnace located a distance of 1 m away. The inside of the furnace is at 1500 K and the intensity escaping from the hole is essentially blackbody intensity at that temperature. (a) What is the radiative heat rate hitting the detector? (b) Assuming that the pyrometer has been calibrated for the situation in (a), what temperature would the pyrometer indicate if the nonabsorbing gas between furnace and detector were replaced by one with an (average) absorption coefficient of $\kappa = 0.1\text{ m}^{-1}$?

Solution

(a) A total radiative intensity of $I_b(T_{\text{furnace}})$ leaves the hole, equally into all directions. From the definition of intensity the total heat rate hitting the detector—assuming the detector to be directly opposite the hole—is

$$\begin{aligned} Q_d &= A_{\text{hole}} I_b(T_f) d\Omega_{\text{hole-detector}} = A_{\text{hole}} I_b(T_f) \frac{A_{\text{detector}}}{S_{\text{hd}}^2} \\ Q_d &= \frac{A_{\text{hole}} \sigma T_f^4 A_{\text{detector}}}{\pi S_{\text{hd}}^2} \\ &= \frac{1\text{ cm}^2 \times 5.670 \times 10^{-12}\text{ W/cm}^2\text{ K}^4 \times 1500^4\text{ K}^4 \times 1\text{ mm}^2}{\pi 1\text{ m}^2} \end{aligned}$$

$$Q_d = 9.137 \times 10^{-6}\text{ W} = 9.137\text{ }\mu\text{W}$$

(b) The signal would now be attenuated by a factor of $\exp(-\kappa S_{\text{hd}})$, or

$$\begin{aligned} Q_d^* &= Q_d \exp(-\kappa S_{\text{hd}}) = 9.137\text{ }\mu\text{W} \times \exp(-0.1\text{ m}^{-1} \times 1\text{ m}) \\ &= 8.2675\text{ }\mu\text{W} \end{aligned}$$

Since the d calibration is $Q_d \propto T_f^4$ we have

$$\frac{Q_d}{T_f^4} = \frac{Q_d^*}{T_f^{*4}} \rightarrow T_f^{*4} = \frac{Q_d^*}{Q_d} T_f^4$$

$$T_f^* = \left(\frac{8.2675}{9.137} \right)^{1/4} 1500 = 1463\text{ K},$$

i.e., the measurement would be wrong by 37 K .

1.18 Consider a pyrometer, which also has a detector area of $1\text{mm} \times 1\text{mm}$, which is black in the wavelength range $1.0\text{ }\mu\text{m} \leq \lambda \leq 1.2\text{ }\mu\text{m}$, and perfectly reflecting elsewhere. In front of the detector is a focussing lens ($f = 10\text{ cm}$) of diameter $D = 2\text{ cm}$, and transmissivity of $\tau_l = 0.9$ (around $1\text{ }\mu\text{m}$). In order to measure the temperature inside a furnace, the pyrometer is focussed onto a hot black surface inside the furnace, a distance of 1 m away from the lens.

(a) How large a spot on the furnace wall does the detector see? (Remember that geometric optics dictates

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}; \quad M = \frac{h(\text{detector size})}{H(\text{spot size})} = \frac{v}{u},$$

where $u = 1\text{ m}$ is the distance from lens to furnace wall, and v is the distance from lens to detector.)

(b) If the temperature of the furnace wall is 1200 K , how much energy is absorbed by the detector per unit time?

(c) It turns out the furnace wall is not really black, but has an emittance of $\epsilon = 0.7$ (around $1\text{ }\mu\text{m}$). Assuming there is no radiation reflected from the furnace surface reaching the detector, what is the true surface temperature for the pyrometer reading of case (b)?

(d) To measure higher temperatures pyrometers are outfitted with filters. If a $\tau_f = 0.7$ filter is placed in front of the lens, what furnace temperature would provide the same pyrometer reading as case (b)?

Solution

(a) From geometric optics

$$\frac{1}{v} = \frac{1}{f} - \frac{1}{u} = \frac{1}{10\text{cm}} - \frac{1}{100\text{cm}} = \frac{9}{100\text{cm}}; \quad v = 11.1\text{cm}$$

$$H = h \frac{u}{v} = 1\text{mm} \times \frac{100}{11.1} = 9\text{mm}.$$

Thus, the spot seen by the pyrometer is $9\text{mm} \times 9\text{mm}$ in size.

(b) Energy leaving spot, intercepted by detector (or, by conservation of energy, energy intercepted by detector, coming from spot) is

$$Q_d = I_{b12}(T)A_H\Omega_{Hl}\tau_l = I_{d12}A_h\Omega_{hl},$$

where $I_{b12}(T) = \int_{\lambda_1}^{\lambda_2} I_{b\lambda}(T)d\lambda$ is emitted intensity from the spot on the furnace wall, I_{d12} is intensity absorbed by the detector, and Ω_{Hl} and Ω_{hl} are the solid angles with which the lens is seen from H and h , respectively. Since H and h are small compared to f and $D = 2R$, we simply evaluate the solid angles from the center of spot H and detector h :

$$\Omega_{Hl} \simeq \frac{\pi R^2}{u^2}; \quad \Omega_{hl} = \frac{\pi R^2}{v^2},$$

and

$$A_H\Omega_{hl} = \pi R^2 \left(\frac{H}{u}\right)^2 = \pi R^2 \left(\frac{h}{v}\right)^2 = A_h\Omega_{hl} = \pi 1\text{cm}^2 \left(\frac{9}{100}\right)^2 = 2.545 \times 10^{-4}\text{cm}^2$$

Therefore, $I_{b12}(T)\tau_l = I_{d12}$, i.e., intensity hitting detector is the *same* as emitted from furnace, except for the attenuation through the lens: total energy is concentrated on a smaller spot by increasing the solid angle. Finally,

$$\begin{aligned} I_{b12} &= \frac{\sigma T^4}{\pi} [f(\lambda_2 T) - f(\lambda_1 T)] = \frac{5.67 \times 10^{-8} \times 1200^4}{\pi} \frac{\text{W}}{\text{m}^2} \underbrace{[f(1440)]}_{0.00961} - \underbrace{[f(1200)]}_{0.00213} \\ &= 280 \frac{\text{W}}{\text{m}^2} \\ &\simeq I_{b\lambda}(1.1\text{ }\mu\text{m}, 1200\text{K})\Delta\lambda = (E_{b\lambda}/T^5)(T^5/\pi)\Delta\lambda = 1.7228 \times 10^{-12} \times \frac{1200^5}{\pi} 0.2 \frac{\text{W}}{\text{m}^2} \\ &= 273 \frac{\text{W}}{\text{m}^2} \end{aligned}$$

and

$$Q_d = 280 \frac{W}{m^2} \times 2.545 \times 10^{-8} m^2 \times 0.9 = 6.41 \mu W$$

(c) The energy hitting detector remains the same and, therefore, so does the intensity emitted from the spot:

$$\epsilon I_{b12}(T_a)(\text{actual}) = I_{b12}(T_p = 1200K)(\text{perceived})$$

or, if we assume the blackbody intensity over the detector range can be approximated by the value at $1.1 \mu m$,

$$\frac{\epsilon}{e^{C_2/\lambda T_a} - 1} \simeq \frac{1}{e^{C_2/\lambda T_p} - 1},$$

leading to

$$\begin{aligned} T_a &= \frac{C_2}{\lambda} \left/ \ln\{1 + \epsilon[e^{C_2/\lambda T_p} - 1]\} \right. \\ &= \frac{14,388 \mu m K}{1.1 \mu m} \left/ \ln\{1 + 0.7[e^{14,388/1.1 \times 1200} - 1]\} \right. \end{aligned}$$

or

$$T_a = 1241K.$$

At these wavelengths Wien's law holds almost exactly, i.e., we may drop the two "1"s from the above equation, and

$$T_a \simeq \frac{C_2}{\lambda} \left/ \ln(\epsilon e^{C_2/\lambda T_p}) \right. = \frac{C_2/\lambda}{\frac{C_2}{\lambda T_p} + \ln \epsilon} = \frac{T_p}{1 + \frac{\lambda T_p}{C_2} \ln \epsilon}$$

which again leads to 1241k.

(d) Similar to (c) we get $\tau_f I_{b12}(T_a) = I_{b12}(T_p)$ and, since $\tau_f = \epsilon_{(c)}$, the answer is the same, i.e.,

$$T_a = 1241K.$$

Thus, a much stronger filter is needed to really extend the pyrometer's range.
