

### Solutions for Chapter 1 problems

**1.1** Displacement from a stable equilibrium position leads to a restoring force, *i.e.* a force that pushes the system back toward the equilibrium position. An example is a pencil resting on a table. If you pick up one end a little, and then release it, the pencil returns to its original position. Displacement from an unstable equilibrium position leads to force that pushes the system away from the equilibrium position. An example is a pencil balanced on its point – a tiny force in any direction causes it to fall over.

**1.2 a)** The electric field due to an infinite sheet of charge is  $E_{sheet} = \frac{\sigma}{2\epsilon_0}$ , where  $\sigma$  is the

charge per unit area (units of C/m<sup>2</sup>). Once the external electric field is applied, our cube has a sheet of positive charge on the left and negative charge on the right; each of these sheets has the same  $\sigma$ , and (within the cube) their fields add. Therefore, the total field inside the cube is

$E = \frac{\sigma}{\epsilon_0}$ . For either of the sheets, the total charge has magnitude

$q = \text{Volume} \times \frac{\text{Charge}}{\text{Volume}} = (\ell^2 x)(ne)$ . Therefore, the charge per area is  $\sigma = xne$ , so  $E = \frac{nex}{\epsilon_0}$ .

**b)** The force exerted on the electrons is

$$(\text{Total charge of electrons}) \frac{\text{Force}}{\text{charge}} = (-ne\ell^3)E = -\frac{n^2 e^2 \ell^3}{\epsilon_0} x.$$

**c)** The above could be written as  $F = -kx$ , with  $k = \frac{n^2 e^2 \ell^3}{\epsilon_0}$ . The angular frequency of

oscillations is  $\omega = \sqrt{k/m}$ . The total mass of the electrons is

$$m = \frac{\text{electrons}}{\text{volume}} \cdot \text{volume} \cdot \frac{\text{mass}}{\text{electron}} = n\ell^3 m_e.$$

$$\text{So, } \omega = \sqrt{\frac{\frac{n^2 e^2 \ell^3}{\epsilon_0}}{n\ell^3 m_e}} = \sqrt{\frac{ne^2}{m_e \epsilon_0}}.$$

**1.3** The equations immediately preceding (1.3.4) are

$$x_0 = A \cos \varphi \quad (1)$$

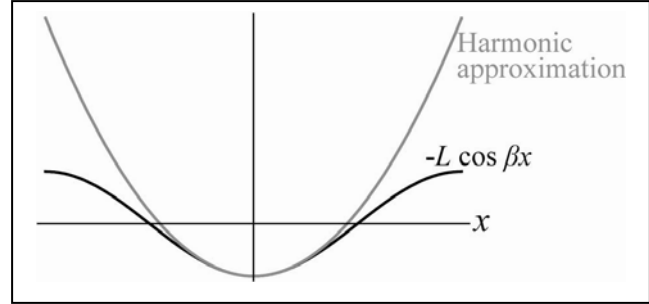
$$\text{and } v_0 = -\omega_0 A \sin \varphi. \quad (2)$$

$$\frac{(2)}{(1)} \rightarrow \frac{v_0}{x_0} = -\omega_0 \tan \varphi \Rightarrow \varphi = \tan^{-1} \left( \frac{-v_0}{\omega_0 x_0} \right), \text{ which is equation (1.3.4b).}$$

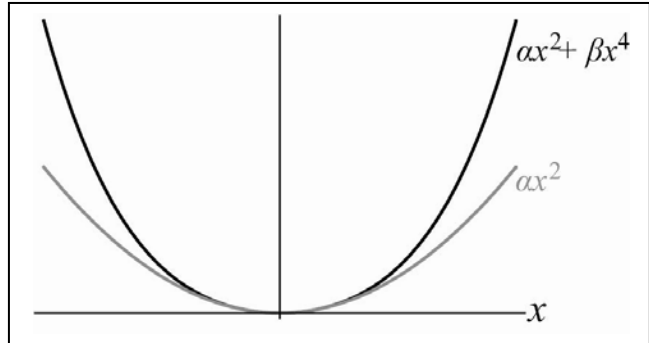
Equation (2) is equivalent to  $\frac{v_0}{\omega_0} = -A \sin \varphi$ . Squaring this, and adding to the square of (1) gives

$$x_0^2 + \frac{v_0^2}{\omega_0^2} = A^2 \cos^2 \varphi + A^2 \sin^2 \varphi = A^2 \Rightarrow A = \sqrt{x_0^2 + \frac{v_0^2}{\omega_0^2}}, \text{ which is equation (1.3.4a).}$$

**1.4** The potential energy of interest is  $U(x) = -L \cos \beta x$ , where  $L$  and  $\beta$  are both  $> 0$ . This has a minimum at  $x = 0$ , and maxima at  $x = \pm \pi / \beta$ . Although this inverted cosine can be well-described by a parabola near the minimum, it flattens out as we move away from the minimum, as shown here. Since  $F = -dU/dx$ , the flattening out corresponds to a decrease in the restoring force as we go to higher amplitudes, rather than the increase that is required to maintain constant period. Therefore, the period increases at higher amplitudes.



**1.5 a)** For small amplitudes, the  $\beta x^4$  term is negligible compared to the  $\alpha x^2$  term, so that  $U \cong \alpha x^2$ . This is of the form  $U = \frac{1}{2} k x^2$ , so the effective spring constant is  $k = 2\alpha$ , and the angular frequency of oscillation is  $\omega = \sqrt{k/m} = \sqrt{2\alpha/m}$ . **b)** Because  $\beta$  is positive, the exact potential energy  $U = \alpha x^2 + \beta x^4$  eventually becomes steeper



than the harmonic approximation  $U \cong \alpha x^2$ , as shown here. Since  $F = -dU/dx$ , the steeper slope corresponds to a stronger restoring force than what is required to keep the period constant. Therefore, the angular frequency increases at large amplitudes.

**1.6** Equation (1.3.3) states  $\omega_0 = \sqrt{k/m}$ , which makes it clear that the angular frequency for a harmonic oscillator does not depend on the amplitude or initial conditions. The seeming contradiction in the rewritten version of (1.3.4a) can be resolved by realizing that  $v_0$ ,  $x_0$ , and  $A$  are interdependent, as shown by equations (1.3.4), leading to cancellations in the expression

$$\omega_0 = \frac{v_0}{\sqrt{A^2 - x_0^2}}. \text{ In other words, as we increase the amplitude } A, \text{ the factors } v_0 \text{ and } \sqrt{A^2 - x_0^2}$$

increase proportionally, so that their ratio remains constant. As example, let's choose  $\varphi = 45^\circ$ .

From (1.3.4b), we have  $\varphi = \tan^{-1}\left(\frac{-v_0}{\omega_0 x_0}\right) \Rightarrow \tan \varphi = \frac{-v_0}{\omega_0 x_0}$ . For our example,  $\tan \varphi = 1$ , so

$v_0 = -\omega_0 x_0$ . Thus, the initial velocity  $v_0$  is proportional to the initial position  $x_0$ . Since  $x = A \cos(\omega t + \varphi)$ , we have  $x_0 = A \cos \varphi$ , which becomes  $x_0 = A/\sqrt{2}$  for our example. Thus, the initial position  $x_0$  is proportional to the amplitude. Since  $v_0$ ,  $x_0$ , and  $A$  are proportional to each other, any change in one of them must be accompanied by a proportional change in the

others; this means that the ratio  $\omega_0 = \frac{v_0}{\sqrt{A^2 - x_0^2}}$  remains constant.

**1.7** We use the trigonometric identity  $\cos(A+B) = \cos A \cos B - \sin A \sin B$  to obtain

$$A \cos(\omega_0 t + \varphi) = A \cos \omega_0 t \cos \varphi - A \sin \omega_0 t \sin \varphi ,$$

This can equal  $A_1 \cos \omega_0 t + A_2 \sin \omega_0 t$  if

$$A_1 = A \cos \varphi \quad (1)$$

$$\text{and } A_2 = -A \sin \varphi . \quad (2)$$

Dividing (2) by (1) gives  $-\tan \varphi = A_2 / A_1 \Rightarrow \varphi = \tan^{-1}(-A_2 / A_1)$ , as required. Taking the combination  $(2)^2 + (1)^2$  gives

$$A_2^2 + A_1^2 = A^2 \sin^2 \varphi + A^2 \cos^2 \varphi = A^2 \Rightarrow A = \sqrt{A_1^2 + A_2^2} ,$$

again as required.

**1.8** The acceleration of the platform is  $\ddot{x} = -A\omega_0^2 \cos \omega_0 t$ , which has a maximum value of  $A\omega_0^2$ . Following the hint, the maximum amplitude for which the mass stays in contact with the platform corresponds to setting this maximum acceleration equal to  $g$ :

$$A\omega_0^2 = g \Leftrightarrow A = g / \omega_0^2 = g / (2\pi f)^2 ,$$

where  $f = 5$  Hz. Plugging in the numbers gives  $A = 0.993$  cm, which rounds to 1 cm.

**1.9** We choose  $t = 0$  such that the position of the finger is

$$x = A \cos \omega t \Rightarrow \dot{x} = -A\omega \sin \omega t \Rightarrow \dot{x}_{\max} = A\omega .$$

The spacing between finger shadows is

$$\begin{aligned} \text{shadow spacing} &= (\text{period of TV flicker}) \dot{x}_{\max} = \frac{1}{f_{TV}} A\omega \Rightarrow f_{TV} = \frac{A\omega}{\text{shadow spacing}} \\ &\Rightarrow f_{TV} = \frac{A2\pi f_{\text{finger}}}{\text{shadow spacing}} . \end{aligned}$$

In my experiment, I used an amplitude of  $A = 5''$  (*i.e.* peak-to-peak amplitude of  $10''$ ), and measured a shadow spacing of  $2''$ . Plugging in the numbers gives  $f_{TV} = 63$  Hz. Since we're told that the frequency is either 30 Hz or 60 Hz, it must be 60 Hz.

**1.10** Read the aside about the arctan function in section 1.3. Explain why a more complete version of equation (1.3.4b) would be  $\varphi = \tan^{-1}\left(\frac{-v_0}{\omega x_0}\right) + \begin{cases} 0 & \text{if } x_0 > 0 \\ \pi & \text{if } x_0 < 0 \end{cases}$ .

**1.10** From the aside about the arctan function in section 1.3, we understand that a calculator or symbolic algebra program returns a value from 0 to  $\pi/2$  for the arctan of a positive argument, and a value from  $-\pi/2$  to 0 for the arctan of a negative argument. From (1.3.4b), we

have  $\varphi = \tan^{-1}\left(\frac{-v_0}{\omega x_0}\right)$ . For any initial  $z$

in quadrant 1 (shown in the figure here), we have  $x_0 > 0$  and  $v_0 < 0$  (because the vector representing  $z$  rotates counterclockwise as time progresses).

Therefore,  $\frac{-v_0}{\omega x_0} > 0$ , and the  $\varphi$  returned

by a calculator is between 0 and  $\pi/2$ ; we can see from the figure that this is correct. However,

for quadrant 2,  $x_0 < 0$  and  $v_0 < 0$ , so that  $\frac{-v_0}{\omega x_0} < 0$ . Therefore, the  $\varphi$  returned by a calculator

is between  $-\pi/2$  and 0, whereas from the figure we see that  $\varphi$  should be between  $\pi/2$  and  $\pi$ .

Therefore, we must add  $\pi$  to the result from the calculator to get the correct value of  $\varphi$ . For

quadrant 3,  $x_0 < 0$  and  $v_0 > 0$ , so that  $\frac{-v_0}{\omega x_0} > 0$ , and a calculator returns a result for  $\varphi$  between

0 and  $\pi/2$ . From the figure, we see that  $\varphi$  should be between  $\pi$  and  $3\pi/2$ , so that we must add  $\pi$  to the result from the calculator. (We could instead subtract  $\pi$  to get a result between  $-\pi/2$

and  $-\pi$ .) Finally, for quadrant 4  $x_0 > 0$  and  $v_0 > 0$ . Therefore,  $\frac{-v_0}{\omega x_0} < 0$  and a calculator

returns a result for  $\varphi$  between 0 and  $-\pi/2$ , which is appropriate.

**1.11 a)**  $F_{Tot} = -\frac{GMm}{x^2} = m\ddot{x} \Rightarrow \ddot{x} + \frac{GM}{x^2} = 0.$

**b)** If  $x_1$  is a solution of this DEQ, then

$$\ddot{x}_1 + \frac{GM}{x_1^2} = 0. \quad (1)$$

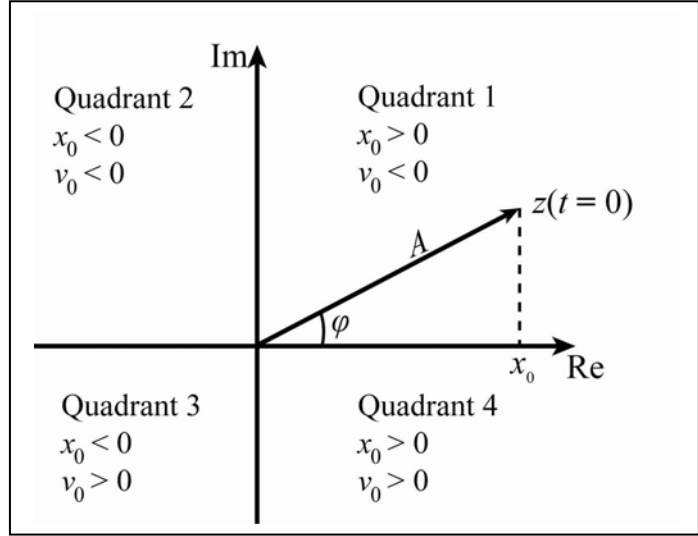
Similarly, if  $x_2$  is a solution, then

$$\ddot{x}_2 + \frac{GM}{x_2^2} = 0. \quad (2)$$

For  $x_1 + x_2$  to be a solution, we would need

$$\frac{d}{dt}(x_1 + x_2) + \frac{GM}{(x_1 + x_2)^2} = 0 \Rightarrow \ddot{x}_1 + \ddot{x}_2 = -\frac{GM}{(x_1 + x_2)^2}.$$

However, by forming the combination (1) + (2), we have



$$\ddot{x}_1 + \ddot{x}_2 + \frac{GM}{x_1^2} + \frac{GM}{x_2^2} = 0 \Rightarrow \ddot{x}_1 + \ddot{x}_2 = -GM \left( \frac{1}{x_1^2} + \frac{1}{x_2^2} \right).$$

So,  $x_1 + x_2$  is only a solution if

$$\frac{1}{(x_1 + x_2)^2} = \frac{1}{x_1^2} + \frac{1}{x_2^2},$$

which is not generally true.

**1.12 a)** As charge is added to a capacitor (positive charge on one plate and negative on the other), it becomes increasingly difficult to add more charge, because it's repelled by the charge already there. Similarly, as we compress a spring, it becomes increasingly difficult to compress it further, because of Hooke's Law:  $F = -kx$ .

**b)** From section 1.5, we have that  $L\ddot{q} = -\frac{1}{C}q$  is isomorphic to  $m\ddot{x} = -kx$ . Therefore,  $L\ddot{q}$ , the voltage across the inductor, is isomorphic to  $m\ddot{x}$ , which equals the total force. Hence voltages are analogous to forces.

The capacitance is defined by  $q = CV \Leftrightarrow V = q/C$ . This means that a capacitor with a large  $C$  can store a large charge  $q$  without developing much voltage. The voltage developed when we apply charge to a capacitor is analogous to the force developed when we apply displacement to a spring (e.g. by compressing it); a large  $k$  results in a large force, whereas a large  $C$  results in a small  $V$ . Therefore, in the isomorphism between the electrical and mechanical oscillators, it is reasonable that  $k$  is isomorphic to  $1/C$  rather than  $C$ .

**1.13** From (1.6.1), the first-order Taylor series approximation is

$$f(x_0 + a) \approx f(x_0) + a \left. \frac{df}{dx} \right|_{x_0}.$$

When expanding around  $x_0 = 0$ , we can set  $a = x$  and write

$$f(x) \approx f(0) + x \left. \frac{df}{dx} \right|_{x=0}.$$

For our case,

$$f(0) = (1+x)^n \Big|_{x=0} = 1 \text{ and } \frac{df}{dx} = n(1+x)^{n-1} \Rightarrow \left. \frac{df}{dx} \right|_{x=0} = n.$$

Therefore,  $f(x) \approx 1 + nx$ , Q.E.D.

**1.14 a), b)** Figure is shown to the right.

**c)** Because the potential energy of the object cannot exceed its total energy, the particle is confined to the range shown. Therefore,  $\beta x$  is

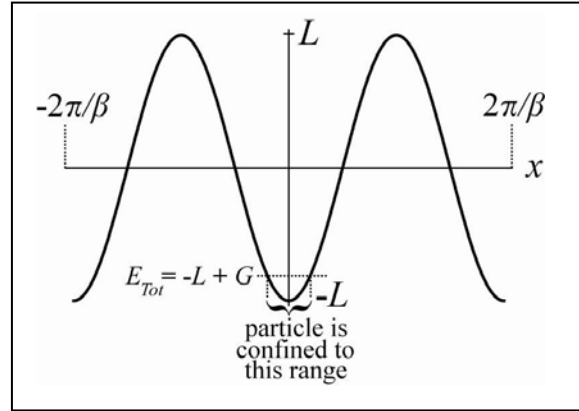
small, so that  $\cos \beta x \cong 1 - \beta^2 x^2 / 2$ , and

$U(x) = -L \cos \beta x \cong -L + L\beta^2 x^2 / 2$ . This has the form  $U = \text{const.} + \frac{1}{2} kx^2$ , with the effective spring

constant given by  $k = L\beta^2$ . The constant term  $-L$  in the potential energy does not affect the

motion. (One can always add any overall constant to the potential energy.) Therefore, the

motion is harmonic, with  $\omega = \sqrt{k/m} = \beta\sqrt{L/m}$ .



**1.15** Any complex number can be expressed in polar form. Let  $C_1 = A_1 e^{i\phi_1}$  and  $C_2 = A_2 e^{i\phi_2}$ .

Then  $\left| \frac{C_1}{C_2} \right| = \left| \frac{A_1 e^{i\phi_1}}{A_2 e^{i\phi_2}} \right| = \left| \frac{A_1 e^{i(\phi_1 - \phi_2)}}{A_2} \right| = \frac{A_1}{A_2}$ , while  $\left| \frac{C_1}{C_2} \right| = \frac{A_1}{A_2}$ . ✓

**1.16 a)**  $8.8 + i 2.2$  (this equals  $9.07e^{i0.245}$ ).

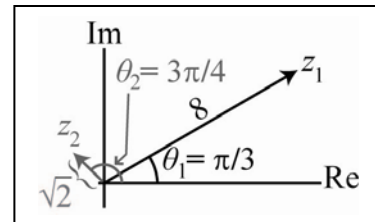
**b)**  $6.1 e^{i1.2} + 1.2 e^{i1.7} = 6.1(\cos 1.2 + i \sin 1.2) + 1.2(\cos 1.7 + i \sin 1.7) = 2.06 + i 6.88$  (This equals  $7.18e^{i1.28}$ .)

**c)**  $(3.2 + i 6.7)(5.6 - i 4.5) = 17.92 - i 14.40 + i 37.52 + 30.15 = 48.07 + i 23.12$  (This equals  $53.34e^{i0.448}$ .)

**d)**  $(6.1 e^{i1.2})(1.2 e^{i1.7}) = (6.1 \cdot 1.2)e^{i(1.2+1.7)} = 7.32e^{i2.9}$  (This equals  $-7.11 + i 1.75$ )

**e)** It's easier to add with the Cartesian representation and to multiply with the polar representation.

**1.17 a)**  $z_1$  is a vector in the complex plane of length 8, with an angle of  $\pi/6 = 30^\circ$  relative to the real axis.  $z_2$  is a vector of length  $\sqrt{2}$  with an angle of  $3\pi/4 = 135^\circ$  relative to the real axis. These are shown here.



**b)** We use the result from section 1.8 that we can write a complex

number either in polar form  $z = Ae^{i\theta}$ , or in Cartesian form

$z = a + ib$ , with  $a = A \cos \theta$  and  $b = A \sin \theta$ . Therefore, the real part of  $z_1$  is

$a_1 = 8 \cos \frac{\pi}{6} = 4\sqrt{3} \cong 6.93$ , and the imaginary part of  $z_1$  is  $b_1 = 8 \sin \frac{\pi}{6} = 4$ . Similarly,

$a_2 = \sqrt{2} \cos \frac{3\pi}{4} = -1$  and  $b_2 = \sqrt{2} \sin \frac{3\pi}{4} = 1$ .

**c)** Using the result from part b, we have

$$z_1 + z_2 = (4\sqrt{3} + i 4) + (-1 + i) = (4\sqrt{3} - 1) + i 5 \cong 5.93 + i 5.$$

Next, we use the result from section 1.8 that we can write a complex number either in polar form

$z = Ae^{i\theta}$ , or in Cartesian form  $z = a + ib$ , with