

Chapter One

Problem 1.1 (i): Expanding the expression

$$F_{ij}\delta_{jk} = F_{i1}\delta_{1k} + F_{i2}\delta_{2k} + F_{i3}\delta_{3k}$$

Of the three terms on the right hand side, only one is nonzero. It is equal to F_{i1} if $k = 1$, F_{i2} if $k = 2$, or F_{i3} if $k = 3$. Thus, it is simply equal to F_{ik} .

Problem 1.1 (ii):

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$$

Problem 1.1 (iii): By Part (i) of this problem, with $F_{ij} = \delta_{ij}$, the result follows immediately.

Problem 1.1 (iv): Using the $\varepsilon - \delta$ identity, we obtain $(\delta_{ij}\delta_{ij} = \delta_{ii} = 3)$

$$\varepsilon_{ijk}\varepsilon_{ijk} = \delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ij} = 9 - 3 = 6$$

Problem 1.1 (v):

$$A_i A_j \varepsilon_{ijk} = -A_i A_j \varepsilon_{jik} = -A_j A_i \varepsilon_{jik} = -A_i A_j \varepsilon_{ijk}$$

where, in going from the third to the fourth expression, subscript i is renamed as j and j is renamed as i . Since the only real number that is equal to its own negative is zero, the result follows.

Problem 1.1 (vi): The result follows by definition. A cyclic permutation of i , j , and k does not change the value. Interchanging of any two subscripts, which is equivalent to permutation in the opposite direction, changes the sign.

Problem 1.2 (i): We have

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) &= [(A_i \hat{\mathbf{e}}_i) \times (B_j \hat{\mathbf{e}}_j)] \times [(C_m \hat{\mathbf{e}}_m) \times (D_n \hat{\mathbf{e}}_n)] \\ &= (A_i B_j \varepsilon_{ijk} \hat{\mathbf{e}}_k) \times (C_m D_n \varepsilon_{mnp} \hat{\mathbf{e}}_p) \\ &= A_i B_j C_m D_n \varepsilon_{ijk} \varepsilon_{mnp} \varepsilon_{kpr} \hat{\mathbf{e}}_r \\ &= A_i B_j C_m D_n \varepsilon_{mnp} (\delta_{ip} \delta_{jr} - \delta_{ir} \delta_{jp}) \hat{\mathbf{e}}_r \\ &= A_p B_r C_m D_n \varepsilon_{mnp} \hat{\mathbf{e}}_r - A_r B_p C_m D_n \varepsilon_{mnp} \hat{\mathbf{e}}_r \end{aligned}$$

where we have used the $\varepsilon - \delta$ identity. Since $B_r \hat{\mathbf{e}}_r = \mathbf{B}$, $C_m D_n \varepsilon_{mnp} A_p = \mathbf{C} \times \mathbf{D} \cdot \mathbf{A}$, and so on, we can write

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = [\mathbf{A} \cdot (\mathbf{C} \times \mathbf{D})]\mathbf{B} - [\mathbf{B} \cdot (\mathbf{C} \times \mathbf{D})]\mathbf{A}$$

Although the above vector identity is established using an orthonormal basis, it holds in a general coordinate system. Also, it can be shown that

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = [\mathbf{D} \cdot (\mathbf{A} \times \mathbf{B})]\mathbf{C} - [\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})]\mathbf{D}$$

Problem 1.2 (ii): We begin with

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= (A_i \hat{\mathbf{e}}_i) \times (B_j \hat{\mathbf{e}}_j) \cdot (C_m \hat{\mathbf{e}}_m) \times (D_n \hat{\mathbf{e}}_n) \\ &= (A_i B_j \varepsilon_{ijk} \hat{\mathbf{e}}_k) \cdot (C_m D_n \varepsilon_{mnp} \hat{\mathbf{e}}_p) \\ &= A_i B_j C_m D_n \varepsilon_{ijk} \varepsilon_{mnp} \delta_{kp} \\ &= A_i B_j C_m D_n \varepsilon_{ijk} \varepsilon_{mnk} \\ &= A_i B_j C_m D_n (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \\ &= A_i B_j C_m D_n \delta_{im} \delta_{jn} - A_i B_j C_m D_n \delta_{in} \delta_{jm} \end{aligned}$$

where we have used the $\varepsilon - \delta$ identity. Since $C_m \delta_{im} = C_i$ (or $A_i \delta_{im} = A_m$), and $A_i C_i = \mathbf{A} \cdot \mathbf{C}$, and so on, we can write

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= A_i B_j C_i D_j - A_i B_j C_j D_i \\ &= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \end{aligned}$$

New Problem 1.1 ♣: _____

Problem: Show that the dot and cross can be interchanged without changing the value in the scalar triple product

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$$

Solution: We have

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} \times \mathbf{C} &= A_i \hat{\mathbf{e}}_i \cdot B_j C_k \varepsilon_{jkm} \hat{\mathbf{e}}_m = A_i B_j C_k \varepsilon_{jkm} \delta_{im} \\ &= A_i B_j C_k \varepsilon_{jki} = A_i B_j C_k \varepsilon_{ijk} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} \end{aligned}$$

Since i, j , and k can be permuted in a cyclic order, it also follows that

$$A_i B_j C_k \varepsilon_{ijk} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A}$$

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{C} \times \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{B} \times \mathbf{C} \cdot \mathbf{A}$$

Problem 1.2 (iii): Using Part (i) of this problem, we can write

$$(\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A}) = [\mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})]\mathbf{C} - [\mathbf{C} \cdot (\mathbf{C} \times \mathbf{A})]\mathbf{B} = [\mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})]\mathbf{C} - 0$$

and then

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \cdot [(\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{B})] &= (\mathbf{A} \times \mathbf{B}) \cdot [\mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})]\mathbf{C} \\ &= [\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}][\mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})] \\ &= [\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}]^2 \end{aligned}$$

where the identities of the New Problem 1.1 are used to arrive at the last step.

Problem 1.2 (iv):

$$\begin{aligned} (\mathbf{AB})^T &= (A_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_jB_{mn}\hat{\mathbf{e}}_m\hat{\mathbf{e}}_n)^T \\ &= A_{ij}B_{mn}(\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\hat{\mathbf{e}}_m\hat{\mathbf{e}}_n)^T \\ &= A_{ij}B_{mn}\hat{\mathbf{e}}_n\hat{\mathbf{e}}_m\hat{\mathbf{e}}_j\hat{\mathbf{e}}_i \\ &= (B_{mn}\hat{\mathbf{e}}_n\hat{\mathbf{e}}_m)(A_{ij}\hat{\mathbf{e}}_j\hat{\mathbf{e}}_i) \\ &= (\mathbf{B})^T(\mathbf{A})^T \end{aligned}$$

Problem 1.3: Let the new coordinate system be the barred coordinate system. We have

$$\begin{aligned} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1 &= \cos 90 = a_{11}, \quad \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 = \cos 0 = a_{12}, \quad \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_3 = \cos 90 = a_{13} \\ \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1 &= -\cos 45 = a_{21}, \quad \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_2 = \cos 90 = a_{22}, \quad \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_3 = \cos 45 = a_{23} \\ \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_1 &= \cos 45 = a_{31}, \quad \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_2 = \cos 90 = a_{32}, \quad \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_3 = \cos 45 = a_{33} \end{aligned}$$

Thus the transformation matrix is

$$[A] = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Problem 1.4: Using the result of Problem 1.2(iv) and $[A]^T = [A]$, we have

$$([B]^T[A][B])^T = ([A][B])^T([B]^T)^T = [B]^T[A]^T[B] = [B]^T[A][B]$$

Problem 1.5 (i): Since

$$|[A][B]| = |[A]||[B]|$$

and $|[A]| \neq 0$ and $|[B]| \neq 0$, it follows that $|[A][B]| \neq 0$. Hence $[A][B]$ is nonsingular.

Problem 1.5 (ii): By definition of an inverse we have

$$([A][B])([A][B])^{-1} = [I]$$

Now premultiplying both sides with the inverse of $[A]$, we obtain

$$[A]^{-1}[A][B]([A][B])^{-1} = [A]^{-1}[I] = [A]^{-1}$$

$$[B]([A][B])^{-1} = [A]^{-1}[I] = [A]^{-1}$$

Next premultiply both sides with the inverse of $[B]$ and obtain

$$[B]^{-1}[B]([A][B])^{-1} = [B]^{-1}[A]^{-1} \quad \text{or} \quad ([A][B])^{-1} = [B]^{-1}[A]^{-1}$$

Problem 1.6 (a): The matrix form of the equations is

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -8 & -5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 2 \\ 2 \end{Bmatrix}$$

Using Cramer's rule we obtain

$$x_1 = \frac{1}{|A|} \begin{vmatrix} 2 & -1 & -1 \\ 2 & 2 & 1 \\ 2 & -8 & -5 \end{vmatrix} = \frac{1}{|A|} [2(-10 + 8) + (-10 - 2) - (-16 - 4)] = \frac{4}{|A|}$$

$$x_2 = \frac{1}{|A|} \begin{vmatrix} 2 & 2 & -1 \\ 1 & 2 & 1 \\ 4 & 2 & -5 \end{vmatrix} = \frac{1}{|A|} [2(-10 - 2) - 2(-5 - 4) - (2 - 8)] = \frac{0}{|A|}$$

$$x_3 = \frac{1}{|A|} \begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & 2 \\ 4 & -8 & 2 \end{vmatrix} = \frac{1}{|A|} [2(4 + 16) + (2 - 8) + 2(-8 - 8)] = \frac{2}{|A|}$$

where the determinant $|A|$ of the coefficient matrix is

$$|A| = \begin{vmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -8 & -5 \end{vmatrix} = 2(-10 + 8) + (-5 - 4) - (-8 - 8) = 3$$

Hence, $x_1 = 4/3$, $x_2 = 0$, and $x_3 = 2/3$.

Problem 1.6 (b): The matrix form of the equations is

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix}$$

Using Cramer's rule we obtain

$$\begin{aligned}
 x_1 &= \frac{1}{|A|} \begin{vmatrix} 1 & -1 & 0 \\ 2 & 4 & -1 \\ 1 & -1 & 2 \end{vmatrix} = \frac{1}{|A|} [(8 - 1) + (4 + 1) - 0] = \frac{12}{|A|} \\
 x_2 &= \frac{1}{|A|} \begin{vmatrix} 2 & 1 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & 2 \end{vmatrix} = \frac{1}{|A|} [2(4 + 1) - (-2 - 0) - 0] = \frac{12}{|A|} \\
 x_3 &= \frac{1}{|A|} \begin{vmatrix} 2 & -1 & 1 \\ -1 & 4 & 2 \\ 0 & -1 & 1 \end{vmatrix} = \frac{1}{|A|} [2(4 + 2) + (-1 - 0) + (1 - 0)] = \frac{12}{|A|}
 \end{aligned}$$

where the determinant $|A|$ of the coefficient matrix is

$$|A| = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 2(8 - 1) + (-2 - 0) - 0 = 12$$

Hence, $x_1 = x_2 = x_3 = 1$.

Problem 1.7: First note the identity

$$\frac{\partial}{\partial x_j} \left[w_i \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] = w_i \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{\partial w_i}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Therefore

$$\begin{aligned}
 - \int_{\Omega} w_i \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) dv &= \int_{\Omega} \frac{\partial w_i}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) dv - \int_{\Omega} \frac{\partial}{\partial x_j} \left[w_i \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] dv \\
 &= \int_{\Omega} \frac{\partial w_i}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) dv - \oint_{\Gamma} w_i n_j \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) ds
 \end{aligned}$$

where Eq. (1.5-2b) is used.

Problem 1.8: Let $\nabla^2 \psi = u$. Then using Eq. (1.5-9) [with $\psi = \varphi$, $f = 1$, and $\varphi = u$] we obtain

$$\int_{\Omega} \varphi \nabla^4 \psi dv = \int_{\Omega} \varphi \nabla^2 u dv = - \int_{\Omega} \nabla \varphi \cdot \nabla u dv + \oint_{\Gamma} \varphi \frac{\partial u}{\partial n} ds$$

Once again using Eq. (1.5-9)

$$- \int_{\Omega} \nabla \varphi \cdot \nabla u dv = \int_{\Omega} u \nabla^2 \varphi dv - \oint_{\Gamma} u \frac{\partial \varphi}{\partial n} ds$$

Thus we have

$$\int_{\Omega} \varphi \nabla^4 \psi dv = \int_{\Omega} \nabla^2 \psi \nabla^2 \varphi dv + \oint_{\Gamma} \varphi \frac{\partial(\nabla^2 \psi)}{\partial n} ds - \oint_{\Gamma} \nabla^2 \psi \frac{\partial \varphi}{\partial n} ds$$

Problem 1.9 (i): We have

$$\begin{aligned}
 (\mathbf{I} \times \mathbf{A}) \cdot \Phi &= (\delta_{ij} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \times A_k \hat{\mathbf{e}}_k) \cdot (\Phi_{mn} \hat{\mathbf{e}}_m \hat{\mathbf{e}}_n) \\
 &= (A_k \varepsilon_{jkl} \hat{\mathbf{e}}_j \hat{\mathbf{e}}_l) \cdot (\Phi_{mn} \hat{\mathbf{e}}_m \hat{\mathbf{e}}_n) \\
 &= A_k \Phi_{mn} \varepsilon_{jkl} \delta_{lm} \hat{\mathbf{e}}_j \hat{\mathbf{e}}_n \\
 &= A_k \Phi_{mn} \varepsilon_{jkm} \hat{\mathbf{e}}_j \hat{\mathbf{e}}_n \\
 &= \mathbf{A} \times \Phi
 \end{aligned}$$

Problem 1.9 (ii): We have

$$\begin{aligned}
 (\Phi \times \mathbf{A})^T &= (\Phi_{ij} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \times A_k \hat{\mathbf{e}}_k)^T \\
 &= (\Phi_{ij} A_k \varepsilon_{jkl} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_l)^T \\
 &= \Phi_{ij} A_k \varepsilon_{jkl} \hat{\mathbf{e}}_l \hat{\mathbf{e}}_i \\
 &= -\mathbf{A} \times \Phi^T
 \end{aligned}$$

Problem 1.10: Suppose that $\{X\} \neq \{0\}$ and the determinant of $[A - \lambda I]$ is not zero. Since the determinant is not zero it has an inverse. Therefore we have

$$\{X\} = [A - \lambda I]^{-1} \{0\} = \{0\}$$

which contradicts the hypothesis that $\{X\} \neq \{0\}$.